

On long term investment optimality

Anatolii A. Puhalskii *

Institute for Problems in Information Transmission

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Abstract

We study the problem of optimal long term investment with a view to beat a benchmark for a diffusion model of asset prices. Two kinds of objectives are considered. One criterion concerns the probability of outperforming the benchmark and seeks either to minimise the decay rate of the probability that a portfolio exceeds the benchmark or to maximise the decay rate that the portfolio falls short. The other criterion concerns the growth rate of the risk-sensitive utility of wealth which has to be either minimised, for a risk-averse investor, or maximised, for a risk-seeking investor. It is assumed that the mean returns and volatilities of the securities are affected by an economic factor, possibly, in a nonlinear fashion. The economic factor and the benchmark are modelled with general Itô differential equations. The results identify optimal portfolios and produce the decay, or growth, rates. The portfolios have the form of time-homogeneous functions of the economic factor. Furthermore, a uniform treatment is given to the out- and under- performance probability optimisation as well as to the risk-averse and risk-seeking portfolio optimisation. It is shown that there exists a portfolio that optimises the decay rates of both the outperformance probability and the underperformance probability. While earlier research on the subject has relied, for the most part, on the techniques of stochastic optimal control and dynamic programming, in this contribution the quantities of interest are studied directly by employing the methods of the large deviation theory. The key to the analysis is to recognise the setup in question as a case of coupled diffusions with time scale separation, with the economic factor representing "the fast motion".

1 Introduction

Recently, two approaches have emerged to constructing long-term optimal portfolios for diffusion models of asset prices: optimising the risk-sensitive criterion and optimising the probability of outperforming a benchmark. In the risk-sensitive framework, one is concerned with the expected utility of wealth $\mathbf{E}e^{\lambda \ln Z_t}$, where Z_t represents the portfolio's wealth at time t and λ is the risk-sensitivity parameter, also referred to as a Hara parameter, which expresses the investor's degree of risk aversion if $\lambda < 0$ or of risk-seeking if $\lambda > 0$. When

*Email: aapuhalski@aim.com

trying to beat the benchmark, Y_t , the expected utility of wealth is given by $\mathbf{E}e^{\lambda \ln(Z_t/Y_t)}$. Since typically those expectations grow, or decay, at an exponential rate with t , one is led to optimise that rate, so an optimal portfolio for the risk-averse investor (respectively, for the risk-seeking investor) is defined as the one that minimises (respectively, maximises) the limit, assuming it exists, of $(1/t) \ln \mathbf{E}e^{\lambda \ln(Z_t/Y_t)}$, as $t \rightarrow \infty$. In a similar vein, there are two ways to define the criterion when the objective is to outperform the benchmark. One can either choose the limit of $(1/t) \ln \mathbf{P}(\ln(Z_t/Y_t) \leq 0)$, as $t \rightarrow \infty$, as the quantity to be minimised or the limit of $(1/t) \ln \mathbf{P}(\ln(Z_t/Y_t) \geq 0)$ as the quantity to be maximised. Arguably, the former criterion is favoured by the risk-averse investor and the latter, by the risk-seeking one. More generally, one may look at the limits of $(1/t) \ln \mathbf{P}(\ln(Z_t/Y_t) \leq q)$ or of $(1/t) \ln \mathbf{P}(\ln(Z_t/Y_t) \geq q)$, for some threshold q .

Risk-sensitive optimisation has received considerable attention in the literature and has been studied under various sets of hypotheses. Bielecki and Pliska [5] consider a setting with constant volatilities and with mean returns of the securities being affine functions of an economic factor, which is modelled as a Gaussian process that satisfies a linear stochastic differential equation with constant diffusion coefficients. For the risk-averse investor, they find an asymptotically optimal portfolio and the long term growth rate of the expected utility of wealth. Subsequent research has relaxed some of the assumptions made, such as the independence of the diffusions driving the economic factor process and the asset price process, see Kuroda and Nagai [24], Bielecki and Pliska [6]. Fleming and Sheu [17], [18] analyse both the risk-averse and the risk-seeking setups. A benchmarked setting is studied by Davis and Lleo [11], [12], [13], the latter two papers being concerned with diffusions with jumps as driving processes. Nagai [29] assumes general mean returns and volatilities and the factor process being the solution to a general stochastic differential equation and obtains an optimal portfolio for the risk-averse investor when there is no benchmark involved. Special one-dimensional models are treated in Fleming and Sheu [16] and Bielecki, Pliska, and Sheu [7]. The methods of the aforementioned papers rely on the tools of stochastic optimal control. A Hamilton–Jacobi–Bellman equation is invoked in order to identify a portfolio that minimises the expected utility of wealth on a finite horizon. Afterwards, a limit is taken as the length of time goes to infinity. The optimal portfolio is expressed in terms of a solution to a Riccati algebraic equation in the affine case, and to an ergodic Bellman equation, in the general case.

The criterion of the probability of outperformance is considered in Pham [31], who studies a one-dimensional benchmarked setup. The minimisation of the underperformance probability for the Bielecki and Pliska [5] model is addressed in Hata, Nagai, and Sheu [20], who look at a no benchmark setup. Nagai [30] studies the general model with the riskless asset as the benchmark. Those authors build on the foundation laid by the work on the risk-sensitive optimisation by applying stochastic control methods in order to identify an optimal risk-sensitive portfolio, first, and, afterwards, use duality considerations to optimise the probabilities of out/under performance. The risk-sensitive optimal portfolio for an appropriately chosen risk-sensitivity parameter is found to be optimal for the out/under performance probability criterion, although a proof of that fact is missing for the general model in Nagai [30]. The parameter is between zero and one for the outperformance case and is negative, for the underperformance case. Puhalskii [32] analyses the out/under performance probabilities directly and obtains a portfolio that is asymptotically optimal both

for the outperformance and underperformance probabilities, the limitation of their study being that it is confined to a geometric Brownian motion model of the asset prices with no economic factor involved. Puhalskii and Stutzer [34] study the underperformance probability for the model in Nagai [30] with a general benchmark by applying direct methods. Their results imply that the portfolio found in Nagai [30] is optimal.

Whereas the cases of a negative Hara parameter for risk-sensitive optimisation and of the underperformance probability minimisation seem to be fairly well understood, the setups of a positive Hara parameter for risk-sensitive optimisation and of the outperformance probability optimisation are lacking clarity. The reason seems to be twofold. Firstly, the expected utility of wealth may grow at an infinite exponential rate for certain $\lambda \in [0, 1]$, see Fleming and Sheu [18]. Secondly, the analysis of the ergodic Bellman equation presents difficulty because no Lyapunov function is readily available, cf., condition (A3) in Kaise and Sheu [22]. Although Pham [31] carries out a detailed study and identifies the threshold value of λ when "the blow-up" occurs for an affine model of one security and one factor, for the multidimensional case, we are unaware of results that produce asymptotically optimal portfolios either for the risk-sensitive criterion with a positive Hara parameter or for maximising the outperformance probability.

The purpose of this paper is to fill in the aforementioned gaps. As in Puhalskii and Stutzer [34], we study the benchmarked version of the general model introduced in Nagai [29], [30]. Capitalising on the insights in Puhalskii and Stutzer [34], we identify an optimal portfolio for maximising the outperformance probability. For the risk-sensitive setup, we prove that there is a threshold value $\bar{\lambda} \in (0, 1]$ such that for all $\lambda < \bar{\lambda}$ there exists an asymptotically optimal risk-seeking portfolio. It is arrived at as an optimal outperformance portfolio for certain threshold q . If $\lambda > \bar{\lambda}$, there is a portfolio such that the expected utility of wealth grows at an infinite exponential rate. Furthermore, we give a uniform treatment to the out- and under- performance probability optimisation as well as to the risk-averse and risk-seeking portfolio optimisation. Not only is that of methodological value, but the proofs for the case of a positive Hara parameter rely on the optimality properties of a portfolio with a negative Hara parameter. We show that the same portfolio optimises both the underperformance and outperformance probabilities, in line with conclusions in Puhalskii [32]. Similarly, the same procedure can be used for finding optimal risk-sensitive portfolios both for the risk-averse investor and for the risk-seeking investor. As in Nagai [29, 30] and Puhalskii and Stutzer [34], the portfolios are expressed in terms of solutions to ergodic Bellman equations.

Since we use the methods of Puhalskii and Stutzer [34], no stochastic control techniques are invoked and standard tools of large deviation theory are employed, such as a change of a probability measure and an exponential Markov inequality. The key is to recognise that one deals with a case of coupled diffusions with time scale separation and introduce the empirical measure of the factor process which is "the fast motion". Another notable feature is an extensive use of the minimax theorem and a characterisation of the optimal portfolios in terms of saddle points. Being more direct than the one based on the stochastic optimal control theory, this approach streamlines considerations, e.g., there is no need to contend with a Hamilton-Jacobi-Bellman equation on finite time, thereby enabling us both to obtain new results and relax or drop altogether a number of assumptions present in the earlier research on the subject. For instance, we do not restrict the class of portfolios under consideration to portfolios whose total wealth is a sublinear function of the economic factor, nor do we

require that the limit growth rate of the expected utility of wealth be an essentially smooth (or "steep") function of the Hara parameter, which conditions are needed in Pham [31] even for a one-dimensional model. On the other hand, when optimizing the underperformance probability and when optimizing the risk-sensitive criterion with a negative Hara parameter, we produce ϵ -asymptotically optimal portfolios, rather than asymptotically optimal portfolios as in Hata, Nagai, and Sheu [20] and in Nagai [30], which distinction does not seem to be of great significance. Besides, our conditions seem to be less restrictive.

The proofs of certain saddle-point properties for positive Hara parameters relying on the associated properties for negative Hara parameters, this paper includes a substantial portion of the developments in Puhalskii and Stutzer [34]. The presentation, however, is self-contained and does not depend on any of the results of Puhalskii and Stutzer [34]. This is how this paper is organised. In Section 2, we define the model and state the main results. In addition, more detail is given on the relation to earlier work. The proofs are provided in Section 4 whereas Section 3 and the appendix are concerned with laying the groundwork and shedding additional light on the model of Pham [31].

2 A model description and main results

We start by recapitulating the setup of Puhalskii and Stutzer [34]. One is concerned with a portfolio consisting of n risky securities priced S_t^1, \dots, S_t^n at time t and a safe security of price S_t^0 at time t . We assume that, for $i = 1, 2, \dots, n$,

$$\frac{dS_t^i}{S_t^i} = a^i(X_t) dt + b^i(X_t)^T dW_t$$

and that

$$\frac{dS_t^0}{S_t^0} = r(X_t) dt,$$

where X_t represents an economic factor. It is governed by the equation

$$dX_t = \theta(X_t) dt + \sigma(X_t) dW_t. \quad (2.1)$$

In the equations above, the $a^i(x)$ are real-valued functions, the $b^i(x)$ are \mathbb{R}^k -valued functions, $\theta(x)$ is an \mathbb{R}^l -valued function, $\sigma(x)$ is an $l \times k$ -matrix, W_t is a k -dimensional standard Wiener process, and $S_0^i > 0$, T is used to denote the transpose of a matrix or a vector. Accordingly, the process $X = (X_t, t \geq 0)$ is l -dimensional.

Benchmark $Y = (Y_t, t \geq 0)$ follows an equation similar to those for the risky securities:

$$\frac{dY_t}{Y_t} = \alpha(X_t) dt + \beta(X_t)^T dW_t,$$

where $\alpha(x)$ is an \mathbb{R} -valued function, $\beta(x)$ is an \mathbb{R}^k -valued function, and $Y_0 > 0$.

All processes are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. It is assumed, furthermore, that the processes $S^i = (S_t^i, t \geq 0)$, X , and $Y = (Y_t, t \geq 0)$ are adapted to (right-continuous) filtration $\mathbf{F} = (\mathcal{F}_t, t \geq 0)$ and that $W = (W_t, t \geq 0)$ is an \mathbf{F} -Wiener process.

We let $a(x)$ denote the n -vector with entries $a^1(x), \dots, a^n(x)$, let $b(x)$ denote the $n \times k$ matrix with rows $b^1(x)^T, \dots, b^n(x)^T$ and let $\mathbf{1}$ denote the n -vector with unit entries. The matrix functions $b(x)b(x)^T$ and $\sigma(x)\sigma(x)^T$ are assumed to be uniformly positive definite and bounded. The functions $a(x)$, $r(x)$, $\theta(x)$, $\alpha(x)$, $b(x)$, $\sigma(x)$, and $\beta(x)$ are assumed to be continuously differentiable with bounded derivatives and the function $\sigma(x)\sigma(x)^T$ is assumed to be twice continuously differentiable. In addition, the following "linear growth" condition is assumed: for some $K > 0$ and all $x \in \mathbb{R}^l$,

$$|a(x)| + |r(x)| + |\alpha(x)| + |\theta(x)| \leq K(1 + |x|).$$

The function $|\beta(x)|^2$ is assumed to be bounded and bounded away from zero. (We will also indicate how the results change if the benchmark "is not volatile" meaning that $\beta(x) = 0$.) Under those hypotheses, the processes S^i , X , and Y are well defined, see, e.g., chapter 5 of Karatzas and Shreve [23].

For the factor process, we assume that

$$\limsup_{|x| \rightarrow \infty} \theta(x)^T \frac{x}{|x|^2} < 0. \quad (2.2)$$

Thus, X has a unique invariant measure, see, e.g., Bogachev, Krylov, and Röckner [9]. As for the initial condition, we will assume that

$$\mathbf{E} e^{\gamma |X_0|^2} < \infty, \text{ for some } \gamma > 0. \quad (2.3)$$

Sometimes it will be required that $|X_0|$ be, moreover, bounded.

The investor holds l_t^i shares of risky security i and l_t^0 shares of the safe security at time t , so the total wealth is given by $Z_t = \sum_{i=1}^n l_t^i S_t^i + l_t^0 S_t^0$. Portfolio $\pi_t = (\pi_t^1, \dots, \pi_t^n)^T$ specifies the proportions of the total wealth invested in the risky securities so that, for $i = 1, 2, \dots, n$, $l_t^i S_t^i = \pi_t^i Z_t$. The processes $\pi^i = (\pi_t^i, t \geq 0)$ are assumed to be $(\mathcal{B} \otimes \mathcal{F}_t, t \geq 0)$ -progressively measurable, where \mathcal{B} denotes the Borel σ -algebra on \mathbb{R}_+ , and such that $\int_0^t \pi_s^{i^2} ds < \infty$ a.s. We do not impose any other restrictions on the magnitudes of the π_t^i so that unlimited borrowing and shortselling are allowed.

Let

$$L_t^\pi = \frac{1}{t} \ln\left(\frac{Z_t}{Y_t}\right).$$

Since the amount of wealth invested in the safe security is $(1 - \sum_{i=1}^n \pi_t^i)Z_t$, in a standard fashion by using the self-financing condition, one obtains that

$$\frac{dZ_t}{Z_t} = \sum_{i=1}^n \pi_t^i \frac{dS_t^i}{S_t^i} + \left(1 - \sum_{i=1}^n \pi_t^i\right) \frac{dS_t^0}{S_t^0}.$$

Assuming that $Z_0 = Y_0$ and letting $c(x) = b(x)b(x)^T$, we have by Itô's lemma that, cf. Pham

[31],

$$L_t^\pi = \frac{1}{t} \int_0^t \left(\pi_s^T a(X_s) + (1 - \pi_s^T \mathbf{1}) r(X_s) - \frac{1}{2} \pi_s^T c(X_s) \pi_s - \alpha(X_s) + \frac{1}{2} |\beta(X_s)|^2 \right) ds + \frac{1}{t} \int_0^t \left(b(X_s)^T \pi_s - \beta(X_s) \right)^T dW_s. \quad (2.4)$$

One can see that L_t^π is "of order one" for t great. Therefore, if one embeds the probability of outperformance $\mathbf{P}(\ln(Z_t/Y_t) \geq 0)$ (respectively, the probability of underperformance $\mathbf{P}(\ln(Z_t/Y_t) \leq 0)$) into the parameterised family of probabilities $\mathbf{P}(L_t^\pi \geq q)$ (respectively, $\mathbf{P}(L_t^\pi \leq q)$), one will concern themselves with large deviation probabilities.

Let, for $u \in \mathbb{R}^n$ and $x \in \mathbb{R}^l$,

$$M(u, x) = u^T (a(x) - r(x) \mathbf{1}) - \frac{1}{2} u^T c(x) u + r(x) - \alpha(x) + \frac{1}{2} |\beta(x)|^2 \quad (2.5a)$$

and

$$N(u, x) = b(x)^T u - \beta(x). \quad (2.5b)$$

A change of variables brings (2.4) to the form

$$L_t^\pi = \int_0^1 M(\pi_{ts}, X_{ts}) ds + \frac{1}{\sqrt{t}} \int_0^1 N(\pi_{ts}, X_{ts})^T dW_s^t, \quad (2.6)$$

where $W_s^t = W_{ts}/\sqrt{t}$. We note that $W^t = (W_s^t, s \in [0, 1])$ is a Wiener process relative to $\mathbf{F}^t = (\mathcal{F}_{ts}, s \in [0, 1])$. The righthand side of (2.6) can be viewed as a diffusion process with a small diffusion coefficient which lives in "normal time" represented by the variable s , whereas in X_{ts} and π_{ts} "time" is accelerated by a factor of t . Furthermore, on introducing $\pi_s^t = \pi_{ts}$, $X_s^t = X_{ts}$, assuming that, for suitable function $u(\cdot)$, $\pi_s^t = u(X_s^t)$, defining

$$\Psi_s^t = \int_0^s M(u(X_{\tilde{s}}^t), X_{\tilde{s}}^t) d\tilde{s} + \frac{1}{\sqrt{t}} \int_0^s N(u(X_{\tilde{s}}^t), X_{\tilde{s}}^t)^T dW_{\tilde{s}}^t, \quad (2.7)$$

so that $L_t^\pi = \Psi_1^t$, and writing (2.1) as

$$X_s^t = X_0^t + t \int_0^s \theta(X_{\tilde{s}}^t) d\tilde{s} + \sqrt{t} \int_0^s \sigma(X_{\tilde{s}}^t) dW_{\tilde{s}}^t, \quad (2.8)$$

one can see that (2.7) and (2.8) make up a similar system of equations to those studied in Liptser [26] and in Puhalskii [33]. The following heuristic derivation which is based on the Large Deviation Principle in Theorem 2.1 in Puhalskii [33] provides insight into our results below.

Let us introduce additional pieces of notation first. Let \mathbb{C}^2 represent the set of real-valued twice continuously differentiable functions on \mathbb{R}^l . For $f \in \mathbb{C}^2$, we let $\nabla f(x)$ represent the gradient of f at x which is regarded as a column l -vector and we let $\nabla^2 f(x)$ represent the $l \times l$ -Hessian matrix of f at x . Let \mathbb{C}_0^1 and \mathbb{C}_0^2 represent the sets of functions of compact support on \mathbb{R}^l that are once and twice continuously differentiable, respectively. Let \mathbb{P} denote the set of probability densities $m = (m(x), x \in \mathbb{R}^l)$ on \mathbb{R}^l such that $\int_{\mathbb{R}^l} |x|^2 m(x) dx < \infty$ and let $\hat{\mathbb{P}}$ denote the set of probability densities m from \mathbb{P} such that $m \in \mathbb{W}_{\text{loc}}^{1,1}(\mathbb{R}^l)$ and $\sqrt{m} \in \mathbb{W}^{1,2}(\mathbb{R}^l)$, where \mathbb{W} is used for denoting a Sobolev space, see, e.g., Adams and Fournier [1]. Let $\mathbb{C}([0, 1], \mathbb{R})$ represent the set of continuous real-valued functions on $[0, 1]$ being endowed with the uniform topology and let $\mathbb{C}_\uparrow([0, 1], \mathbb{M}(\mathbb{R}^l))$ represent the set of functions μ_t on $[0, 1]$ with values in the set $\mathbb{M}(\mathbb{R}^l)$ of (nonnegative) measures on \mathbb{R}^l such that $\mu_t(\mathbb{R}^l) = t$ and $\mu_t - \mu_s$ is a nonnegative measure when $t \geq s$. The space $\mathbb{M}(\mathbb{R}^l)$ is assumed to be equipped with the weak topology and the space $\mathbb{C}_\uparrow([0, 1], \mathbb{M}(\mathbb{R}^l))$, with the uniform topology. Let the empirical process of $X^t = (X_s^t, s \in [0, 1])$, which is denoted by $\mu^t = (\mu^t(ds, dx))$, be defined by the equation

$$\mu^t([0, s], \Gamma) = \int_0^s \chi_\Gamma(X_{t\tilde{s}}) d\tilde{s},$$

with Γ denoting a Borel subset of \mathbb{R}^l and with $\chi_\Gamma(x)$ representing the indicator function of Γ . We note that both X^t and $\pi^t = (\pi_s^t, s \in [0, 1])$ are \mathbf{F}^t -adapted.

If one were to apply to the processes $\Psi^t = (\Psi_s^t, s \in [0, 1])$ and μ^t Theorem 2.1 in Puhalskii [33], then the pair (Ψ^t, μ^t) would satisfy the Large Deviation Principle in $\mathbb{C}([0, 1]) \times \mathbb{C}_\uparrow([0, 1], \mathbb{M}_1(\mathbb{R}^l))$, as $t \rightarrow \infty$, with the deviation function (usually referred to as a rate function)

$$\begin{aligned} \mathbf{J}(\Psi, \mu) = & \int_0^1 \sup_{\lambda \in \mathbb{R}} \left(\lambda (\dot{\Psi}_s - \int_{\mathbb{R}^l} M(u(x), x) m_s(x) dx) - \frac{1}{2} \lambda^2 \int_{\mathbb{R}^l} |N(u(x), x)|^2 m_s(x) dx \right. \\ & + \sup_{f \in \mathbb{C}_0^1(\mathbb{R}^l)} \int_{\mathbb{R}^l} \left(\nabla f(x)^T \left(\frac{1}{2} \operatorname{div} (\sigma(x) \sigma(x)^T m_s(x)) - (\theta(x) + \lambda \sigma(x)^T N(u(x), x)) m_s(x) \right) \right. \\ & \left. \left. - \frac{1}{2} |\sigma(x)^T \nabla f(x)|^2 m_s(x) \right) dx \right) ds, \quad (2.9) \end{aligned}$$

provided the function $\Psi = (\Psi_s, s \in [0, 1])$ is absolutely continuous w.r.t. Lebesgue measure on \mathbb{R} and the function $\mu = (\mu_s(\Gamma))$, when considered as a measure on $[0, 1] \times \mathbb{R}^l$, is absolutely continuous w.r.t. Lebesgue measure on $\mathbb{R} \times \mathbb{R}^l$, i.e., $\mu(ds, dx) = m_s(x) dx ds$, where $m_s(x)$, as a function of x , belongs to $\hat{\mathbb{P}}$ for almost all s . If those conditions do not hold, then $\mathbf{J}(\Psi, \mu) = \infty$. (We assume that the divergence of a square matrix is evaluated rowwise.)

Integration by parts yields an alternative form:

$$\mathbf{J}(\Psi, \mu) = \int_0^1 \sup_{\lambda \in \mathbb{R}} \left(\lambda (\dot{\Psi}_s - \int_{\mathbb{R}^l} M(u(x), x) m_s(x) dx) - \frac{1}{2} \lambda^2 \int_{\mathbb{R}^l} |N(u(x), x)|^2 m_s(x) dx \right.$$

$$\begin{aligned}
& + \sup_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} \left(-\frac{1}{2} \operatorname{tr} (\sigma(x) \sigma(x)^T \nabla^2 f(x)) - \nabla f(x)^T (\theta(x) + \lambda \sigma(x)^T N(u(x), x)) \right. \\
& \quad \left. - \frac{1}{2} |\sigma(x)^T \nabla f(x)|^2 \right) m_s(x) dx \Big) ds, \quad (2.10)
\end{aligned}$$

with $\operatorname{tr} \Sigma$ standing for the trace of square matrix Σ . Since $L_t^\pi = \Psi_1^t$, by projection, L_t^π obeys the large deviation principle in \mathbb{R} for rate t with the deviation function $\mathbf{I}(L) = \inf\{\mathbf{J}(\Psi, \mu) : \Psi_1 = L\}$. Therefore,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L_t^\pi \geq q) \leq - \inf_{L \geq q} \mathbf{I}(L). \quad (2.11)$$

The integrand against ds in (2.10) being a convex function of $\dot{\Psi}_s$ and of $m_s(x)$, along with the requirements that $\int_0^1 \dot{\Psi}_s ds = L$ and $\int_{\mathbb{R}^l} m_s(x) dx = 1$ imply, by Jensen's inequality, that one may assume that $\dot{\Psi}_s = L$ and that $m_s(x)$ does not depend on s either, so that $m_s(x) = m(x)$. Hence,

$$\begin{aligned}
\inf_{L \geq q} \mathbf{I}(L) &= \inf_{L \geq q} \inf_{m \in \hat{\mathbb{P}}} \sup_{\lambda \in \mathbb{R}} \left(\lambda \left(L - \int_{\mathbb{R}^l} M(u(x), x) m(x) dx \right) - \frac{1}{2} \lambda^2 \int_{\mathbb{R}^l} |N(u(x), x)|^2 m(x) dx \right. \\
& \quad + \sup_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} \left(-\frac{1}{2} \operatorname{tr} (\sigma(x) \sigma(x)^T \nabla^2 f(x)) - \nabla f(x)^T (\theta(x) + \lambda \sigma(x)^T N(u(x), x)) \right. \\
& \quad \left. \left. - \frac{1}{2} |\sigma(x)^T \nabla f(x)|^2 \right) m(x) dx \right).
\end{aligned}$$

On noting that the expression on the righthand side is convex in (L, m) and is concave in (λ, f) , one hopes to be able to apply a minimax theorem to change the order of taking inf and sup so that

$$\begin{aligned}
\inf_{L \geq q} \mathbf{I}(L) &= \sup_{\lambda \in \mathbb{R}} \sup_{f \in \mathbb{C}_0^2} \inf_{L \geq q} \inf_{m \in \hat{\mathbb{P}}} \left(\lambda \left(L - \int_{\mathbb{R}^l} M(u(x), x) m(x) dx \right) - \frac{1}{2} \lambda^2 \int_{\mathbb{R}^l} |N(u(x), x)|^2 m(x) dx \right. \\
& \quad + \int_{\mathbb{R}^l} \left(-\frac{1}{2} \operatorname{tr} (\sigma(x) \sigma(x)^T \nabla^2 f(x)) - \nabla f(x)^T (\theta(x) + \lambda \sigma(x)^T N(u(x), x)) \right. \\
& \quad \left. \left. - \frac{1}{2} |\sigma(x)^T \nabla f(x)|^2 \right) m(x) dx \right). \quad (2.12)
\end{aligned}$$

If $\lambda < 0$, then the infimum over $L \geq q$ equals $-\infty$. If $\lambda \geq 0$, it is attained at $L = q$ and $\inf_{m \in \hat{\mathbb{P}}}$ "is attained at a δ -density" so that (2.12) results in

$$\begin{aligned}
\inf_{L \geq q} \mathbf{I}(L) &= \sup_{\lambda \in \mathbb{R}_+} \sup_{f \in \mathbb{C}_0^2} \left(\lambda q - \sup_{x \in \mathbb{R}^l} (\lambda M(u(x), x) + \frac{1}{2} \lambda^2 |N(u(x), x)|^2 \right. \\
& \quad \left. + \frac{1}{2} \operatorname{tr} (\sigma(x) \sigma(x)^T \nabla^2 f(x)) + \nabla f(x)^T (\theta(x) + \lambda \sigma(x)^T N(u(x), x)) + \frac{1}{2} |\sigma(x)^T \nabla f(x)|^2) \right). \quad (2.13)
\end{aligned}$$

For an optimal outperforming portfolio, one wants to maximise the righthand side of (2.11) over functions $u(x)$, so the righthand side of (2.13) has to be minimised. Assuming one can apply minimax considerations once again yields

$$\inf_{u(\cdot)} \inf_{L \geq q} \mathbf{I}(L) = \sup_{\lambda \in \mathbb{R}_+} \sup_{f \in \mathbb{C}_0^2} \left(\lambda q - \sup_{x \in \mathbb{R}^l} \sup_{u \in \mathbb{R}^n} \left(\lambda M(u, x) + \frac{1}{2} \lambda^2 |N(u, x)|^2 \right. \right. \\ \left. \left. + \nabla f(x)^T (\theta(x) + \lambda \sigma(x)^T N(u, x)) \right) + \frac{1}{2} \text{tr} (\sigma(x) \sigma(x)^T \nabla^2 f(x)) + \frac{1}{2} |\sigma(x)^T \nabla f(x)|^2 \right).$$

By (2.5a) and (2.5b), the $\sup_{u \in \mathbb{R}^n} = \infty$ if $\lambda > 1$ so, on recalling (2.11), it is reasonable to conjecture that

$$\sup_{\pi} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L_t^\pi \geq q) = - \sup_{\lambda \in [0, 1]} \sup_{f \in \mathbb{C}_0^2} \left(\lambda q - \sup_{x \in \mathbb{R}^l} \sup_{u \in \mathbb{R}^n} \left(\lambda M(u, x) + \frac{1}{2} \lambda^2 |N(u, x)|^2 \right. \right. \\ \left. \left. + \nabla f(x)^T (\theta(x) + \lambda \sigma(x)^T N(u, x)) \right) + \frac{1}{2} \text{tr} (\sigma(x) \sigma(x)^T \nabla^2 f(x)) + \frac{1}{2} |\sigma(x)^T \nabla f(x)|^2 \right) \quad (2.14)$$

and an optimal portfolio is of the form $u(X_t)$, with $u(x)$ attaining the supremum with respect to u on the righthand side of (2.14) for λ and f that deliver their respective suprema. Similar arguments may be applied to finding $\inf_{\pi} \liminf_{t \rightarrow \infty} (1/t) \ln \mathbf{P}(L_t^\pi < q)$. Unfortunately, we are unable to fill in the gaps in the above deduction, e.g., in order for the results of Puhalskii [33] to apply, the function $u(x)$ has to be bounded in x , while the optimal portfolio typically is not. Besides, it is not at all obvious that the optimal portfolio should be expressed as a function of the economic factor. Nevertheless, the above line of reasoning is essentially correct, as our main results show. Besides, there is a special case which we analyse at the final stages of our proofs that allows a direct application of Theorem 2.1 in Puhalskii [33]. We now proceed to stating the results. That requires introducing more pieces of notation and providing background information.

The following nondegeneracy condition is needed. (It was introduced in Puhalskii and Stutzer [34].) Let I_k denote the $k \times k$ -identity matrix and let

$$Q_1(x) = I_k - b(x)^T c(x)^{-1} b(x).$$

The matrix $Q_1(x)$ represents the orthogonal projection operator onto the null space of $b(x)$ in \mathbb{R}^k . We will assume that

- (N) 1. The matrix $\sigma(x) Q_1(x) \sigma(x)^T$ is uniformly positive definite.
2. The quantity $\beta(x)^T Q_2(x) \beta(x)$ is bounded away from zero, where

$$Q_2(x) = Q_1(x) (I_k - \sigma(x)^T (\sigma(x) Q_1(x) \sigma(x)^T)^{-1} \sigma(x)) Q_1(x). \quad (2.15)$$

Condition (N) admits the following geometric interpretation.

Lemma 2.1. *The matrix $\sigma(x) Q_1(x) \sigma(x)^T$ is uniformly positive definite if and only if arbitrary nonzero vectors from the ranges of $\sigma(x)^T$ and $b(x)^T$, respectively, are at angles bounded away from zero if and only if the matrix $c(x) - b(x) \sigma(x)^T (\sigma(x) \sigma(x)^T)^{-1} \sigma(x) b(x)^T$ is uniformly*

positive definite. Also, $\beta(x)^T Q_2(x) \beta(x)$ is bounded away from zero if and only if the projection of $\beta(x)$ onto the null space of $b(x)$ is of length bounded away from zero and is at angles bounded away from zero to all projections onto that null space of nonzero vectors from the range of $\sigma(x)^T$.

The proof of the lemma is provided in the appendix. Under part 1 of condition (N), we have that $k \geq n + l$ and the rows of the matrices $\sigma(x)$ and $b(x)$ are linearly independent. Part 2 of condition (N) implies that $\beta(x)$ does not belong to the sum of the ranges of $b(x)^T$ and of $\sigma(x)^T$. (Indeed, if that were the case, then $Q_1(x)\beta(x)$, which is the projection of $\beta(x)$ onto the null space of $b(x)$, would also be the projection of a vector from the range of $\sigma(x)^T$ onto the null space of $b(x)$.) Thus, $k > n + l$.

The righthand side of (2.14) motivates the following definitions. Let, given $x \in \mathbb{R}^l$, $\lambda \in \mathbb{R}$, and $p \in \mathbb{R}^l$,

$$\check{H}(x; \lambda, p) = \lambda \sup_{u \in \mathbb{R}^n} (M(u, x) + \frac{1}{2} \lambda |N(u, x)|^2 + p^T \sigma(x) N(u, x)) + p^T \theta(x) + \frac{1}{2} |\sigma(x)^T p|^2. \quad (2.16)$$

By (2.5a) and (2.5b), the latter righthand side is finite if $\lambda < 1$, with the supremum being attained at

$$u(x) = \frac{1}{1 - \lambda} c(x)^{-1} (a(x) - r(x) \mathbf{1} - \lambda b(x) \beta(x) + b(x) \sigma(x)^T p). \quad (2.17)$$

Furthermore,

$$\begin{aligned} & \sup_{u \in \mathbb{R}^n} (M(u, x) + \frac{1}{2} \lambda |N(u, x)|^2 + p^T \sigma(x) N(u, x)) \\ &= \frac{1}{2} \frac{1}{1 - \lambda} \|a(x) - r(x) \mathbf{1} - \lambda b(x) \beta(x) + b(x) \sigma(x)^T p\|_{c(x)^{-1}}^2 \\ & \quad + \frac{1}{2} \lambda |\beta(x)|^2 + r(x) - \alpha(x) + \frac{1}{2} |\beta(x)|^2 - \beta(x)^T \sigma(x)^T p, \end{aligned} \quad (2.18)$$

where, for $y \in \mathbb{R}^n$ and positive definite symmetric $n \times n$ -matrix Σ , we denote $\|y\|_\Sigma^2 = y^T \Sigma y$.

Therefore, on introducing

$$T_\lambda(x) = \sigma(x) \sigma(x)^T + \frac{\lambda}{1 - \lambda} \sigma(x) b(x)^T c(x)^{-1} b(x) \sigma(x)^T, \quad (2.19a)$$

$$S_\lambda(x) = \frac{\lambda}{1 - \lambda} (a(x) - r(x) \mathbf{1} - \lambda b(x) \beta(x))^T c(x)^{-1} b(x) \sigma(x)^T - \lambda \beta(x)^T \sigma(x)^T + \theta(x)^T, \quad (2.19b)$$

and

$$\begin{aligned} R_\lambda(x) &= \frac{\lambda}{2(1 - \lambda)} \|a(x) - r(x) \mathbf{1} - \lambda b(x) \beta(x)\|_{c(x)^{-1}}^2 + \lambda(r(x) - \alpha(x) + \frac{1}{2} |\beta(x)|^2) \\ & \quad + \frac{1}{2} \lambda^2 |\beta(x)|^2, \end{aligned} \quad (2.19c)$$

we have that

$$\check{H}(x; \lambda, p) = \frac{1}{2} p^T T_\lambda(x) p + S_\lambda(x) p + R_\lambda(x). \quad (2.20)$$

Let us note that, by condition (N), $T_\lambda(x)$ is a uniformly positive definite matrix.

If $\lambda = 1$, then, on noting that

$$\begin{aligned} M(u, x) + \frac{1}{2} |N(u, x)|^2 + p^T \sigma(x) N(u, x) &= u^T (a(x) - r(x) \mathbf{1} - b(x) \beta(x) + b(x) \sigma(x)^T p) \\ &\quad + r(x) - \alpha(x) + |\beta(x)|^2 - p^T \sigma(x) \beta(x), \end{aligned} \quad (2.21)$$

we have that $\check{H}(x; 1, p) < \infty$ if and only if

$$a(x) - r(x) \mathbf{1} - b(x) \beta(x) + b(x) \sigma(x)^T p = 0, \quad (2.22)$$

in which case

$$\check{H}(x; 1, p) = r(x) - \alpha(x) + |\beta(x)|^2 - p^T \sigma(x) \beta(x) + p^T \theta(x) + \frac{1}{2} |\sigma(x)^T p|^2. \quad (2.23)$$

As mentioned, if $\lambda > 1$, then the righthand side of (2.16) equals infinity. Consequently, $\check{H}(x; \lambda, p)$ is a lower semicontinuous function of (λ, p) with values in $\mathbb{R} \cup \{+\infty\}$. By Lemma 3.5 below, $\check{H}(x; \lambda, p)$ is convex in (λ, p) .

We define, given $f \in \mathbb{C}^2$,

$$H(x; \lambda, f) = \check{H}(x; \lambda, \nabla f(x)) + \frac{1}{2} \text{tr}(\sigma(x) \sigma(x)^T \nabla^2 f(x)). \quad (2.24)$$

By the convexity of \check{H} , the function $H(x; \lambda, f)$ is convex in (λ, f) .

Let

$$F(\lambda) = \inf_{f \in \mathbb{C}^2} \sup_{x \in \mathbb{R}^l} H(x; \lambda, f) \text{ if } \lambda < 1, \quad (2.25)$$

$F(1) = \lim_{\lambda \uparrow 1} F(\lambda)$, $F(\lambda) = \infty$ if $\lambda > 1$, and

$$\bar{\lambda} = \sup\{\lambda \in \mathbb{R} : F(\lambda) < \infty\}.$$

By $H(x; \lambda, f)$ being convex in (λ, f) , $F(\lambda)$ is convex for $\lambda < 1$, so $F(1)$ is well defined, see, e.g., Theorem 7.5 on p.57 in Rockafellar [35]. The function $F(\lambda)$ is seen to be convex as a function on \mathbb{R} . It is finite when $\lambda < \lambda_0$, for some $\lambda_0 \in (0, 1]$, which is obtained by taking $f(x) = \kappa |x|^2$, $\kappa > 0$ being small enough (see Lemma 3.1 for more detail). Therefore $\bar{\lambda} \in (0, 1]$. Lemma 3.2 below establishes that $F(0) = 0$, that $F(\lambda)$ is lower semicontinuous on \mathbb{R} and that if $F(\lambda)$ is finite, with $\lambda < 1$, then the infimum in (2.25) is attained at function f^λ which satisfies the equation

$$H(x; \lambda, f^\lambda) = F(\lambda), \text{ for all } x \in \mathbb{R}^l. \quad (2.26)$$

Furthermore, $f^\lambda \in \mathbb{C}_\ell^1$, with \mathbb{C}_ℓ^1 representing the set of real-valued continuously differentiable functions on \mathbb{R}^l whose gradients satisfy the linear growth condition. Thus, the infimum in (2.25) can be taken over $\mathbb{C}^2 \cap \mathbb{C}_\ell^1$ when $\lambda < 1$. Equation (2.26) is dubbed an ergodic Bellman

equation, see, e.g., Fleming and Sheu [18], Kaise and Sheu [22], Hata, Nagai, and Sheu [20], Ichihara [21].

Let \mathcal{P} represent the set of probability measures ν on \mathbb{R}^l such that $\int_{\mathbb{R}^l} |x|^2 \nu(dx) < \infty$. For $\nu \in \mathcal{P}$, we let $\mathbb{L}^2(\mathbb{R}^l, \mathbb{R}^l, \nu(dx))$ represent the Hilbert space (of the equivalence classes) of \mathbb{R}^l -valued functions $h(x)$ on \mathbb{R}^l that are square integrable with respect to $\nu(dx)$ equipped with the norm $(\int_{\mathbb{R}^l} |h(x)|^2 \nu(dx))^{1/2}$ and we let $\mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \nu(dx))$ represent the closure in $\mathbb{L}^2(\mathbb{R}^l, \mathbb{R}^l, \nu(dx))$ of the set of gradients of \mathbb{C}_0^1 -functions. We will retain the notation ∇f for the elements of $\mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \nu(dx))$, although those functions might not be proper gradients. Let \mathcal{U}_λ denote the set of functions $f \in \mathbb{C}^2 \cap \mathbb{C}_\ell^1$ such that $\sup_{x \in \mathbb{R}^l} H(x; \lambda, f) < \infty$. The set \mathcal{U}_λ is nonempty if and only if $F(\lambda) < \infty$. It is convenient to write (2.25) in the form, cf. (2.12),

$$F(\lambda) = \inf_{f \in \mathcal{U}_\lambda} \sup_{\nu \in \mathcal{P}} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx), \quad \text{if } \lambda < 1, \quad (2.27)$$

the latter integral possibly being equal to $-\infty$. We adopt the convention that $\inf_\emptyset = \infty$, so that (2.27) holds when $\mathcal{U}_\lambda = \emptyset$ too. Let \mathbb{C}_b^2 represent the subset of \mathbb{C}^2 of functions with bounded second derivatives. Let, for $f \in \mathbb{C}_b^2$ and $m \in \mathbb{P}$,

$$G(\lambda, f, m) = \int_{\mathbb{R}^l} H(x; \lambda, f) m(x) dx. \quad (2.28)$$

This function is well defined, is convex in (λ, f) and is concave in m . By Lemma 3.5 and Lemma 3.6 below, for $\lambda < \bar{\lambda}$, $F(\lambda) = \sup_{m \in \hat{\mathbb{P}}} \inf_{f \in \mathbb{C}_0^2} G(\lambda, f, m)$. One can replace $\hat{\mathbb{P}}$ with \mathbb{P} in the preceding sup and replace \mathbb{C}_0^2 with \mathbb{C}_b^2 in the preceding inf. If $m \in \hat{\mathbb{P}}$, then integration by parts in (2.28) obtains that, for $f \in \mathbb{C}_b^2$,

$$G(\lambda, f, m) = \check{G}(\lambda, \nabla f, m), \quad (2.29)$$

where

$$\check{G}(\lambda, \nabla f, m) = \int_{\mathbb{R}^l} \left(\check{H}(x; \lambda, \nabla f(x)) - \frac{1}{2} \nabla f(x)^T \frac{\text{div}(\sigma(x) \sigma(x)^T m(x))}{m(x)} \right) m(x) dx. \quad (2.30)$$

(Unless specifically mentioned otherwise, it is assumed throughout that $0/0 = 0$. More detail on the integration by parts is given in the proof of Lemma 3.2.) The function $\check{G}(\lambda, \nabla f, m)$ is convex in (λ, f) and is concave in m . The righthand side of (2.30) being well defined for $\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)$, we adopt (2.30) as the definition of $\check{G}(\lambda, \nabla f, m)$ for $(\lambda, \nabla f, m) \in \mathbb{R} \times \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx) \times \hat{\mathbb{P}}$.

Let, for $m \in \hat{\mathbb{P}}$,

$$\check{F}(\lambda, m) = \inf_{\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)} \check{G}(\lambda, \nabla f, m), \quad (2.31)$$

when $\lambda \leq 1$ and let $\check{F}(\lambda, m) = \infty$, for $\lambda > 1$. By Lemma 3.5 below, the infimum in (2.31) is attained uniquely, if finite, the latter always being the case for $\lambda < 1$. Furthermore, if $\lambda < 1$, then $\check{F}(\lambda, m) = \inf_{f \in \mathbb{C}_0^2} G(\lambda, f, m)$. By (2.30), the function $\check{F}(\lambda, m)$ is convex in λ and is concave in m . It is lower semicontinuous in λ and is strictly convex on $(-\infty, 1)$

by Lemma 3.5, so, by convexity, see Corollary 7.5.1 on p.57 in Rockafellar [35], $\check{F}(1, m) = \lim_{\lambda \uparrow 1} \inf_{f \in \mathbb{C}_0^2} G(\lambda, f, m)$. By Lemma 3.6 below, $\lambda q - \check{F}(\lambda, m)$ has saddle point $(\hat{\lambda}, \hat{m})$ in $(-\infty, \bar{\lambda}] \times \hat{\mathbb{P}}$, with $\hat{\lambda}$ being specified uniquely, and the supremum of $\lambda q - F(\lambda)$ over \mathbb{R} is attained at $\hat{\lambda}$.

If $\hat{\lambda} < 1$, which is "the regular case", then \hat{m} is specified uniquely and there exists $\hat{f} \in \mathbb{C}^2 \cap \mathbb{C}_\ell^1$ such that $(\hat{\lambda}, \hat{f}, \hat{m})$ is a saddle point of the function $\lambda q - \check{G}(\lambda, \nabla f, m)$ in $\mathbb{R} \times (\mathbb{C}^2 \cap \mathbb{C}_\ell^1) \times \hat{\mathbb{P}}$, with $\nabla \hat{f}$ being specified uniquely. As a matter of fact, $\hat{f} = f^{\hat{\lambda}}$, so the function \hat{f} satisfies the ergodic Bellman equation

$$H(x; \hat{\lambda}, \hat{f}) = F(\hat{\lambda}), \text{ for all } x \in \mathbb{R}^l. \quad (2.32)$$

The density \hat{m} is the invariant density of a diffusion process in that

$$\int_{\mathbb{R}^l} (\nabla h(x)^T (\hat{\lambda} \sigma(x) N(\hat{u}(x), x) + \theta(x) + \sigma(x) \sigma(x)^T \nabla \hat{f}(x)) + \frac{1}{2} \text{tr}(\sigma(x) \sigma(x)^T \nabla^2 h(x))) \hat{m}(x) dx = 0, \quad (2.33)$$

for all $h \in \mathbb{C}_0^2$. Essentially, equations (2.32) and (2.33) represent Euler–Lagrange equations for $\check{G}(\hat{\lambda}, \nabla f, m)$ at (\hat{f}, \hat{m}) . They specify $\nabla \hat{f}$ and \hat{m} uniquely and imply that (\hat{f}, \hat{m}) is a saddle point of $\check{G}(\hat{\lambda}, \nabla f, m)$, cf., Proposition 1.6 on p.169 in Ekeland and Temam [14]. We define $\hat{u}(x)$ as the u that attains supremum in (2.16) for $\lambda = \hat{\lambda}$ and $p = \nabla \hat{f}(x)$ so that, by (2.17),

$$\hat{u}(x) = \frac{1}{1 - \hat{\lambda}} c(x)^{-1} (a(x) - r(x) \mathbf{1} - \hat{\lambda} b(x) \beta(x) + b(x) \sigma(x)^T \nabla \hat{f}(x)). \quad (2.34)$$

Suppose that $\hat{\lambda} = 1$, which is "the degenerate case". Necessarily, $\bar{\lambda} = 1$, so, the infimum on the righthand side of (2.31) for $\lambda = 1$ and $m = \hat{m}$ is finite and is attained at unique $\nabla \hat{f}$ (see Lemma 3.5). Consequently, $F(1) < \infty$. According to Lemma 3.6 below, cf., (2.22) and (2.33),

$$a(x) - r(x) \mathbf{1} - b(x) \beta(x) + b(x) \sigma(x)^T \nabla \hat{f}(x) = 0 \quad \hat{m}(x) dx\text{-a.e.} \quad (2.35)$$

and

$$\int_{\mathbb{R}^l} (\nabla h(x)^T (-\sigma(x) \beta(x) + \theta(x) + \sigma(x) \sigma(x)^T \nabla \hat{f}(x)) + \frac{1}{2} \text{tr}(\sigma(x) \sigma(x)^T \nabla^2 h(x))) \hat{m}(x) dx = 0, \quad (2.36)$$

provided that $h \in \mathbb{C}_0^2$ and $b(x) \sigma(x)^T \nabla h(x) = 0$ $\hat{m}(x) dx$ -a.e. By (2.21), the value of the expression in the supremum in (2.16) does not depend on the choice of u when $\lambda = 1$ and $p = \nabla \hat{f}(x)$, so, there is some leeway as to the choice of an optimal control. As the concave function $\lambda q - \check{F}(\lambda, \hat{m})$ attains maximum at $\lambda = 1$, $d/d\lambda \check{F}(\lambda, \hat{m}) \Big|_{1-} \leq q$, with $d/d\lambda \check{F}(\lambda, \hat{m}) \Big|_{1-}$ standing for the lefthand derivative of $\check{F}(\lambda, \hat{m})$ at $\lambda = 1$. Hence, there exists bounded continuous function $\hat{v}(x)$ with values in the range of $b(x)^T$ such that $|\hat{v}(x)|^2/2 = q - d/d\lambda \check{F}(\lambda, \hat{m}) \Big|_{1-}$. (For instance, one can take

$\hat{v}(x) = b(x)^T c(x)^{-1/2} z \sqrt{2(q - d/d\lambda \check{F}(\lambda, \hat{m})|_{1-})}$, where z represents an element of \mathbb{R}^n of length one.) We let $\hat{u}(x) = c(x)^{-1} b(x)(\beta(x) + \hat{v}(x))$.

In either case, we define $\hat{\pi}_t = \hat{u}(X_t)$ and, given $\rho > 0$, $\hat{\pi}_t^\rho = \hat{u}^\rho(X_t)$, where $\hat{u}^\rho(x) = \hat{u}(x)\chi_{[0,\rho]}(|x|)$. We introduce, given $\lambda \in \mathbb{R}$, $f \in \mathbb{C}^2$, $m \in \mathbb{P}$, and measurable \mathbb{R}^n -valued function $v = (v(x), x \in \mathbb{R}^l)$,

$$\begin{aligned} \overline{H}(x; \lambda, f, v) = & \lambda M(v(x), x) + \frac{1}{2} |\lambda N(v(x), x) + \sigma(x)^T \nabla f(x)|^2 + \nabla f(x)^T \theta(x) \\ & + \frac{1}{2} \text{tr}(\sigma(x)\sigma(x)^T \nabla^2 f(x)). \end{aligned} \quad (2.37)$$

By (2.16), (2.24), (2.28), (2.34), and (2.37), if $\hat{\lambda} < 1$, then

$$F(\hat{\lambda}) = H(x; \hat{\lambda}, \hat{f}) = \overline{H}(x; \hat{\lambda}, \hat{f}, \hat{u}) = \inf_{f \in \mathbb{C}^2} \sup_{x \in \mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}). \quad (2.38)$$

Let

$$J_q = \sup_{\lambda \leq 1} (\lambda q - F(\lambda)), \quad (2.39a)$$

$$J_q^o = \sup_{\lambda \in [0,1]} (\lambda q - F(\lambda)), \quad (2.39b)$$

and

$$J_q^s = \sup_{\lambda \leq 0} (\lambda q - F(\lambda)). \quad (2.39c)$$

It is noteworthy that if $\hat{\lambda} < 0$, then $J_q^s > 0$ and $J_q^o = 0$, while if $\hat{\lambda} > 0$, then $J_q^o > 0$ and $J_q^s = 0$.

We are in a position to state the first limit theorem.

Theorem 2.1. 1. For arbitrary portfolio $\pi = (\pi_t, t \geq 0)$,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L_t^\pi < q) \geq -J_q^s. \quad (2.40)$$

If, in addition, $|X_0|$ is bounded and $f^\lambda(x)$ is bounded below by an affine function of x when $0 < \lambda < \bar{\lambda}$, then

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L_t^\pi \geq q) \leq -J_q^o. \quad (2.41)$$

2. The following asymptotic bound holds:

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L_t^{\hat{\pi}} > q) \geq -J_q^o. \quad (2.42)$$

If, in addition,

$$\limsup_{\rho \rightarrow \infty} \inf_{f \in \mathbb{C}^2} \sup_{x \in \mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho) \leq F(\hat{\lambda}) \quad (2.43)$$

when $\hat{\lambda} < 0$, then

$$\limsup_{\rho \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L_t^{\hat{\pi}^\rho} \leq q) \leq -J_q^s. \quad (2.44)$$

Remark 2.1. The upper bounds in (2.41) and in (2.44) are of interest only if $\hat{\lambda} > 0$ and $\hat{\lambda} < 0$, respectively.

Remark 2.2. The assertions of Theorem 2.1 hold in the case where $\beta(x) = 0$ too, provided $\inf_{x \in \mathbb{R}^l} r(x) < q$. If $\inf_{x \in \mathbb{R}^l} r(x) \geq q$, then investing in the safe security only is obviously optimal.

Remark 2.3. The requirement that $f^\lambda(x)$ be bounded below by an affine function when $0 < \lambda < \bar{\lambda}$ is fulfilled for the Gaussian model, as we discuss below.

A sufficient condition for (2.43) to hold is given by the next lemma which features a condition introduced by Nagai [30], see also Puhalskii and Stutzer [34]. The proof is relegated to the appendix.

Lemma 2.2. *Suppose that there exist $\varrho > 0$, $C_1 > 0$ and $C_2 > 0$ such that, for all $x \in \mathbb{R}^l$,*

$$(1 + \varrho) \|b(x)\sigma(x)^T \nabla \hat{f}(x)\|_{c(x)^{-1}}^2 - \|a(x) - r(x)\mathbf{1}\|_{c(x)^{-1}}^2 \leq C_1|x| + C_2. \quad (2.45)$$

Then (2.43) holds for $\hat{\lambda} < 0$.

Remark 2.4. As the proof shows, an upper bound on the righthand side of (2.45) can be allowed to grow at a subquadratic rate.

Remark 2.5. The inequality in (2.45) holds provided

$$\limsup_{|x| \rightarrow \infty} \frac{1}{|x|^2} (\|b(x)\sigma(x)^T \nabla \hat{f}(x)\|_{c(x)^{-1}}^2 - \|a(x) - r(x)\mathbf{1}\|_{c(x)^{-1}}^2) < 0. \quad (2.46)$$

It holds also if $b(x)\sigma(x)^T = 0$ which means that the Wiener processes effectively driving the security prices and the economic factor process are independent.

The following theorem shows that the portfolio $\hat{\pi} = (\hat{\pi}_t, t \geq 0)$ is risk-sensitive optimal for suitable q . If F is subdifferentiable at λ , we let $u^\lambda(x)$ represent the function $\hat{u}(x)$ for a value of q that is a subgradient of F at λ . We also let $u^{\lambda, \rho}(x) = u^\lambda(x)\chi_{[0, \rho]}(|x|)$, $\pi_t^\lambda = u^\lambda(X_t)$, $\pi_t^{\lambda, \rho} = u^{\lambda, \rho}(X_t)$, $\pi^\lambda = (\pi_t^\lambda, t \geq 0)$, and $\pi^{\lambda, \rho} = (\pi_t^{\lambda, \rho}, t \geq 0)$. The function F is subdifferentiable at $\lambda < \bar{\lambda}$. It might not be subdifferentiable at $\bar{\lambda}$.

Theorem 2.2. *1. If $0 < \lambda < \bar{\lambda}$, if the function $f^{\lambda(1+\epsilon)}(x)$ is bounded below by an affine function of x when ϵ is small enough, and if $|X_0|$ is bounded, then, for any portfolio $\pi = (\pi_t, t \geq 0)$,*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{E} e^{\lambda t L_t^\pi} \leq F(\lambda).$$

If either $0 < \lambda < \bar{\lambda}$ or $\lambda = \bar{\lambda}$ and F is subdifferentiable at $\bar{\lambda}$, then

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{E} e^{\lambda t L_t^{\pi^\lambda}} \geq F(\lambda).$$

If either $\lambda = \bar{\lambda}$ and F is not subdifferentiable at $\bar{\lambda}$ or $\lambda > \bar{\lambda}$, then there exists portfolio π^λ such that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{E} e^{\lambda t L_t^{\pi^\lambda}} \geq F(\lambda).$$

2. If $\lambda < 0$, then, for any portfolio $\pi = (\pi_t, t \in \mathbb{R}_+)$,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{E} e^{\lambda L_t^\pi} \geq F(\lambda)$$

and, provided (2.43) holds with $\hat{\lambda} = \lambda$ and $\hat{u}^\rho = u^{\lambda, \rho}$ and $|X_0|$ is bounded,

$$\lim_{\rho \rightarrow \infty} \liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{E} e^{\lambda L_t^{\pi^{\lambda, \rho}}} = \lim_{\rho \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{E} e^{\lambda L_t^{\pi^{\lambda, \rho}}} = F(\lambda).$$

Remark 2.6. We recall that $F(\lambda) = \infty$ if $\lambda > \bar{\lambda}$. For a one-dimensional model, $\bar{\lambda}$ is found explicitly in Pham [31], also, see the appendix below. We conjecture that F is differentiable and strictly convex for $\lambda < \bar{\lambda}$, which would imply that π^λ is specified uniquely. This is provably the case for the model of Pham [31] and provided $\lambda < 0$, see Pham [31] and Puhalskii and Stutzer [34], respectively.

If we assume that the functions $a(x)$, $r(x)$, $\alpha(x)$ and $\theta(x)$ are affine functions of x and that the diffusion coefficients are constant, then fairly explicit formulas are available. More specifically, let

$$a(x) = A_1 x + a_2, \quad (2.47a)$$

$$r(x) = r_1^T x + r_2, \quad (2.47b)$$

$$\alpha(x) = \alpha_1^T x + \alpha_2, \quad (2.47c)$$

$$\theta(x) = \Theta_1 x + \theta_2, \quad (2.47d)$$

and

$$b(x) = b, \quad \beta(x) = \beta, \quad \sigma(x) = \sigma, \quad (2.47e)$$

where $A_1 \in \mathbb{R}^{n \times l}$, $a_2 \in \mathbb{R}^n$, $r_1 \in \mathbb{R}^l$, $r_2 \in \mathbb{R}$, $\alpha_1 \in \mathbb{R}^l$, $\alpha_2 \in \mathbb{R}$, Θ_1 is a negative definite $l \times l$ -matrix, $\theta_1 \in \mathbb{R}^l$, b is an $n \times k$ -matrix such that the matrix bb^T is positive definite, β is a non-zero k -vector, and σ is an $l \times k$ -matrix such that the matrix $\sigma\sigma^T$ is positive definite. Condition (N) expresses the requirement that the ranges of σ^T and b^T have the trivial intersection and that β is not an element of the sum of those ranges.

Finding the optimal portfolio $\hat{\pi}_t$ may be reduced to solving an algebraic Riccati equation. We introduce, for $\lambda < 1$,

$$A(\lambda) = \Theta_1 + \frac{\lambda}{1 - \lambda} \sigma b^T c^{-1} (A_1 - \mathbf{1} r_1^T),$$

$$B(\lambda) = T_\lambda(x) = \sigma \sigma^T + \frac{\lambda}{1 - \lambda} \sigma b^T c^{-1} b \sigma^T,$$

and

$$C = \|A_1 - \mathbf{1} r_1^T\|_{c^{-1}}^2.$$

Let us suppose that there exists symmetric $l \times l$ -matrix $P_1(\lambda)$ that satisfies the algebraic Riccati equation

$$P_1(\lambda)B(\lambda)P_1(\lambda) + A(\lambda)^T P_1(\lambda) + P_1(\lambda)A(\lambda) + \frac{\lambda}{1-\lambda} C = 0. \quad (2.48)$$

Conditions for the existence of solutions can be found in Fleming and Sheu [18], see also Willems [41] and Wonham [42]. According to Lemma 3.3 in Fleming and Sheu [18], provided that $\lambda < 0$, there exists unique $P_1(\lambda)$ solving (2.48) such that $P_1(\lambda)$ is negative semidefinite. Furthermore, the matrix

$$D(\lambda) = A(\lambda) + B(\lambda)P_1(\lambda) \quad (2.49)$$

is stable. If $0 < \lambda < 1$ and $F(\lambda) < \infty$, then, by Lemma 4.3 in Fleming and Sheu [18], there exists unique $P_1(\lambda)$ solving (2.48) such that $P_1(\lambda)$ is positive semidefinite and $D(\lambda)$ is semistable. By Theorem 4.6 in Fleming and Sheu [18], the matrix $D(\lambda)$ is stable if λ is small enough.

With $D(\lambda)$ being stable, the equation

$$D(\lambda)^T p_2(\lambda) + E(\lambda) = 0 \quad (2.50)$$

has a unique solution for $p_2(\lambda)$, where

$$E(\lambda) = \frac{\lambda}{1-\lambda} (A_1 - \mathbf{1}r_1^T + b\sigma^T P_1(\lambda))^T c^{-1}(a_2 - r_2\mathbf{1} - \lambda b\beta) + \lambda(r_1 - \alpha_1 - P_1(\lambda)\sigma\beta) + P_1(\lambda)\theta_2. \quad (2.51)$$

Substitution shows that $H(x; \lambda, \tilde{f}^\lambda)$, with $\tilde{f}^\lambda(x) = x^T P_1(\lambda)x/2 + p_2(\lambda)^T x$, does not depend on x . Let m^λ denote the invariant distribution of the linear diffusion

$$dY_t = D(\lambda)Y_t dt + \left(\frac{\lambda}{1-\lambda} \sigma b^T c^{-1}(a_2 - r_2\mathbf{1} - \lambda b\beta + b\sigma^T p_2(\lambda)) - \lambda\sigma\beta + \sigma\sigma^T p_2(\lambda) + \theta_2 \right) dt + \sigma dW_t. \quad (2.52)$$

Then the pair $(\tilde{f}^\lambda, m^\lambda)$ is a saddle point of $\check{G}(\lambda, \nabla f, m)$ as well as of $G(\lambda, f, m)$ considered as functions of $(f, m) \in \mathcal{U}_\lambda \times \hat{\mathbb{P}}$. Hence,

$$\begin{aligned} H(x; \lambda, f^\lambda) &= \check{G}(\lambda, \nabla f^\lambda, m^\lambda) = \inf_{f \in \mathcal{U}_\lambda} \sup_{m \in \hat{\mathbb{P}}} \check{G}(\lambda, \nabla f, m) = \inf_{f \in \mathcal{U}_\lambda} \sup_{m \in \mathbb{P}} G(\lambda, f, m) \\ &= \inf_{f \in \mathcal{U}_\lambda} \sup_{x \in \mathbb{R}^l} H(x; \lambda, f) = F(\lambda), \end{aligned}$$

so \tilde{f}^λ satisfies the Bellman equation (2.26). As a result, under the hypotheses of Fleming and Sheu [18], \tilde{f}^λ is bounded below by an affine function when $\hat{\lambda} \in (0, 1)$. Condition (2.46) is implied by the condition that the matrix $(b\sigma^T P_1(\hat{\lambda}))^T c^{-1} b\sigma^T P_1(\hat{\lambda}) - (A_1 - \mathbf{1}r_1^T)^T c^{-1} (A_1 - \mathbf{1}r_1^T)$ is negative definite.

Furthermore, one can see that

$$\begin{aligned} F(\lambda) &= \frac{1}{2} \|p_2(\lambda)\|_{\sigma\sigma^T}^2 + \frac{1}{2} \frac{\lambda}{1-\lambda} \|a_2 - r_2\mathbf{1} - \lambda b\beta + b\sigma^T p_2(\lambda)\|_{c^{-1}}^2 \\ &\quad + (-\lambda\beta^T \sigma^T + \theta_2^T) p_2(\lambda) + \lambda(r_2 - \alpha_2 + \frac{1}{2} |\beta|^2) + \frac{1}{2} \lambda^2 |\beta|^2 + \frac{1}{2} \text{tr}(\sigma\sigma^T P_1(\lambda)). \end{aligned} \quad (2.53)$$

If $\hat{\lambda} < 1$, equation (2.34) is as follows

$$\hat{u}(x) = \frac{1}{1 - \hat{\lambda}} c^{-1} (A_1 - \mathbf{1} r_1^T + b \sigma^T P_1(\hat{\lambda})) x + \frac{1}{1 - \hat{\lambda}} c^{-1} (a_2 - r_2 \mathbf{1} - \hat{\lambda} b \beta + b \sigma^T p_2(\hat{\lambda})) .$$

and $J_q = F(\hat{\lambda})$. If $\hat{\lambda} = 1$, then one may look, once again, for $\hat{f}(x) = x^T P_1(1)x/2 + p_2(1)^T x$. Substitution in (2.35) yields

$$A_1 - \mathbf{1} r_1^T + b \sigma^T P_1(1) = 0 , \quad (2.54a)$$

$$a_2 - r_2 \mathbf{1} - b \beta + b \sigma^T p_2(1) = 0 . \quad (2.54b)$$

(One can also obtain (2.54a) by multiplying (2.48) through with $1 - \lambda$ and taking a formal limit as $\lambda \uparrow 1$.) If those conditions hold, choosing $\hat{f}(x)$ quadratic is justified. An optimal control is $\hat{u}(x) = c^{-1}(b\beta + \hat{v})$, with \hat{v} coming from the range of b^T and with $|\hat{v}|^2/2 = q - d/d\lambda \check{F}(\lambda, \hat{m}) \Big|_{1-}$.

With $\tilde{\lambda}$ representing the supremum of λ such that $P_1(\lambda)$ exists and $D(\lambda)$ is stable, one has that $\tilde{\lambda} \leq \bar{\lambda}$. Pham [31] shows that, in the one-dimensional case, under broad assumptions, $\tilde{\lambda} = \bar{\lambda}$ and $F(\lambda)$ is differentiable on $(-\infty, \bar{\lambda})$, both cases that $\bar{\lambda} < 1$ and $\bar{\lambda} = 1$ being realisable. The hypotheses in Pham [31], however, rule out the possibility that $\hat{\lambda} = 1$. In the appendix, we complete the analysis of Pham [31] so that the case where $\hat{\lambda} = 1$ is realised too.

Bounds (2.40) and (2.44) of Theorem 2.1 are available in Puhalskii and Stutzer [34] who use a different definition of $H(x; \lambda, f)$. They also assume a more general stability condition than in (2.2) for (2.40) and provide more detail on the relation to earlier results for the underperformance probability optimisation. Theorem 2.1 improves on the results in Puhalskii [32] by doing away with a certain growth requirement on $|\pi_t|$ (see (2.12) in Puhalskii [32]). Maximising the probability of outperformance for a one-dimensional model is studied in Pham [31], who, however, stops short of proving the asymptotic optimality of $\hat{\pi}$ and produces nearly optimal portfolios instead. Besides, the requirements in Pham [31] amount to $F(\lambda)$ being essentially smooth, the portfolio's wealth growing no faster than linearly with the economic factor (see condition in (2.5) in Pham [31]) and $\theta_2 = 0$. On the other hand, it is not assumed in Pham [31] that β does not belong to the sum of the ranges of b^T and σ^T , which property is required by our condition (N).

Most of the results on the risk-sensitive optimisation concern the case of a negative Hara parameter. Theorem 4.1 in Nagai [29] obtains asymptotic optimality of $\pi(\lambda)$, rather than asymptotic ϵ -optimality, for a nonbenchmarked setup under a number of additional conditions, e.g., the interest rate is bounded and the following version of (2.45) is required: $\|b(x)\sigma(x)^T \nabla \hat{f}(x)\|_{c(x)^{-1}}^2 - \|a(x) - r(x)\mathbf{1}\|_{c(x)^{-1}}^2 \rightarrow -\infty$, as $|x| \rightarrow \infty$. (Unfortunately, there are pieces of undefined notation such as $u(0, x; T)$.) Affine models are considered in Bielecki and Pliska [5], [6], Kuroda and Nagai [24], for the nonbenchmarked case, and Davis and Lleo [11], for the benchmarked case. Fleming and Sheu [17], [18] allow λ to assume either sign. Although the latter authors correctly identify the limit quantity in Theorem 2.2 as the righthand side of an ergodic Bellman equation, they prove neither that $F(\lambda)$ is the limit of $(1/t) \ln \mathbf{E} e^{\lambda t L_t^{\pi^\lambda}}$ nor that $F(\lambda)$ is an asymptotic bound for an arbitrary portfolio. Rather, they prove that $F(\lambda)$ can be obtained as the limit of the optimal growth rates associated

with bounded portfolios as the bound constraint is being relaxed. They also require that λ be sufficiently small, if positive. The assertion of part 1 of Theorem 2.2 has not been available in this generality even for the affine model, Theorem 4.1 in Pham [31] tackling a case of one security.

There is another notable distinction of our results. It concerns the stability condition (2.2) on the economic factor process. In some of the literature, similar conditions involve both the parameters of the factor process and of the security price process. For the general model in Nagai [30], it is of the form $\limsup_{|x| \rightarrow \infty} (\theta(x) - \sigma(x)b(x)^T c(x)^{-1}(a(x) - r(x)\mathbf{1}))^T / |x|^2 < 0$, for the Gaussian model in Hata, Nagai, and Sheu [20], it is required that the matrix $\Theta_1 - \sigma b^T c^{-1} A_1$ be stable. It appears as though that imposing a stability condition on the factor process only is more in line with the logic of the model. A similar form of the stability condition to ours appears in Fleming and Sheu [18].

3 Technical preliminaries

In this section, we lay the groundwork for the proofs of the main results. Drawing on Bonnans and Shapiro [10] (see p.14 there), we will say that function $h : \mathbb{T} \rightarrow \mathbb{R}$, with \mathbb{T} representing a topological space, is inf-compact (respectively, sup-compact) if the sets $\{x \in \mathbb{T} : h(x) \leq \delta\}$ (respectively, the sets $\{x \in \mathbb{T} : h(x) \geq \delta\}$) are compact for all $\delta \in \mathbb{R}$. (It is worth noting that Aubin [3] and Aubin and Ekeland [4] adopt a slightly different terminology by requiring only that the sets $\{x \in \mathbb{T} : h(x) \leq \delta\}$ be relatively compact in order for h to be inf-compact. Both definitions are equivalent if h is, in addition, lower semicontinuous.)

We endow the set \mathcal{P} of probability measures ν on \mathbb{R}^l such that $\int_{\mathbb{R}^l} |x|^2 \nu(dx) < \infty$ with the Kantorovich–Rubinstein distance

$$d_1(\mu, \nu) = \sup \left\{ \left| \int_{\mathbb{R}^l} g(x) \mu(dx) - \int_{\mathbb{R}^l} g(x) \nu(dx) \right| : \frac{|g(x) - g(y)|}{|x - y|} \leq 1 \text{ for all } x \neq y \right\}.$$

Convergence with respect to d_1 is equivalent to weak convergence coupled with convergence of first moments, see, e.g., Villani [40]. For $\kappa > 0$, let $f_\kappa(x) = \kappa|x|^2/2$, where $\kappa > 0$ and $x \in \mathbb{R}^l$, and let \mathcal{A}_κ represent the convex hull of \mathbb{C}_0^2 and of the function f_κ .

Lemma 3.1. *There exist $\kappa_0 > 0$ and $\lambda_0 > 0$ such that if $\kappa \leq \kappa_0$ and $\lambda \leq \lambda_0$, then the functions $\int_{\mathbb{R}^l} H(x; \lambda, f_\kappa) \nu(dx)$ and $\inf_{f \in \mathcal{A}_\kappa} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx)$ are sup-compact in $\nu \in \mathcal{P}$ for the Kantorovich–Rubinstein distance d_1 .*

Proof. By (2.20) and (2.24), for $\lambda < 1$,

$$H(x; \lambda, f_\kappa) = \frac{\kappa^2}{2} x^T T_\lambda(x) x + \kappa S_\lambda(x) x + R_\lambda(x) + \text{tr}(\sigma(x) \sigma(x)^T).$$

By (2.2), (2.19a), (2.19b), and (2.19c), as $|x| \rightarrow \infty$, if κ is small, then the dominating term in $(\kappa^2/2) x^T T_\lambda(x) x$ is of order $\kappa^2|x|^2$, the dominating terms in $\kappa S_\lambda(x) x$ are of orders $(\lambda/(1-\lambda)) \kappa|x|^2$ and $-\kappa|x|^2$, and the dominating term in $R_\lambda(x)$ is of order $(\lambda/(1-\lambda)) |x|^2$. If κ is small enough, then $-\kappa|x|^2$ dominates $\kappa^2|x|^2$. For those κ , $(\lambda/(1-\lambda)) |x|^2$ is dominated

by $-\kappa|x|^2$ if λ is small relative to κ . We conclude that, provided κ is small enough, there exist $\lambda_0 > 0$, K_1 , and $K_2 > 0$, such that

$$H(x; \lambda, f_\kappa) \leq K_1 - K_2|x|^2, \quad (3.1)$$

for all $\lambda \leq \lambda_0$. Therefore, given $\delta \in \mathbb{R}$, $\sup_{\nu \in \Gamma_\delta} \int_{\mathbb{R}^l} |x|^2 \nu(dx) < \infty$, where $\Gamma_\delta = \{\nu : \int_{\mathbb{R}^l} H(x; \lambda, f_\kappa) \nu(dx) \geq \delta\}$. In addition, by $H(x; \lambda, f_\kappa)$ being continuous in x and Fatou's lemma, $\int_{\mathbb{R}^l} H(x; \lambda, f_\kappa) \nu(dx)$ is an upper semicontinuous function of ν , so Γ_δ is a closed set. Thus, by Prohorov's theorem, Γ_δ is compact. If $f \in \mathcal{A}_\kappa$, then, in view of Fatou's lemma, (2.20), (2.24), and (3.1), the function $\int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx)$ is upper semicontinuous in ν . Since $f_\kappa \in \mathcal{A}_\kappa$, we obtain that $\inf_{f \in \mathcal{A}_\kappa} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx)$ is sup-compact. \square

Lemma 3.2. *If $\lambda < 1$ and $F(\lambda) < \infty$, then the infimum in (2.25) is attained at \mathbb{C}^2 -function f^λ that satisfies the Bellman equation (2.26) and belongs to \mathbb{C}_ℓ^1 . In addition, the function $F(\lambda)$ is lower semicontinuous and $F(0) = 0$.*

Proof. Let us assume that $F(\lambda) > -\infty$. Applying the reasoning on pp.289–294 in Kaise and Sheu [22], one can see that, for arbitrary $\epsilon > 0$, there exists \mathbb{C}^2 -function f_ϵ such that, for all $x \in \mathbb{R}^l$, $H(x; \lambda, f_\epsilon) = F(\lambda) + \epsilon$. Considering that some details are omitted in Kaise and Sheu [22], we give an outline of the proof, following the lead of Ichihara [21]. As $F(\lambda) < \infty$, by (2.25), there exists function $f_\epsilon^{(1)} \in \mathbb{C}^2$ such that $H(x; \lambda, f_\epsilon^{(1)}) < F(\lambda) + \epsilon$ for all x . Given open ball S , centred at the origin, by Theorem 6.14 on p.107 in Gilbarg and Trudinger [19], there exists \mathbb{C}^2 -solution $f_\epsilon^{(2)}$ to the linear elliptic boundary value problem $H(x; \lambda, f) - (1/2)\nabla f(x)^T T_\lambda(x) \nabla f(x) = F(\lambda) + 2\epsilon$ when $x \in S$ and $f(x) = f_\kappa(x)$ when $x \in \partial S$, with ∂S standing for the boundary of S . Therefore, $H(x; \lambda, f_\epsilon^{(2)}) > F(\lambda) + \epsilon$ in S . By Theorem 8.4 on p.302 of Chapter 4 in Ladyzhenskaya and Ural'tseva [25], for any ball S' contained in S and centred at the origin, there exists \mathbb{C}^2 -solution $f_{\epsilon, S'}^{(3)}$ to the boundary value problem $H(x; \lambda, f) = F(\lambda) + \epsilon$ in S' and $f(x) = f_\kappa(x)$ on $\partial S'$. Since $f_{\epsilon, S'}^{(3)}$ solves the boundary value problem $(1/2)\text{tr}(\sigma(x)\sigma(x)^T \nabla^2 f(x)) = -\check{H}(x; \lambda, \nabla f_{\epsilon, S'}^{(3)}(x)) + F(\lambda) + \epsilon$ when $x \in S'$ and $f(x) = f_\kappa(x)$ when $x \in \partial S'$, we have by Theorem 6.17 on p.109 of Gilbarg and Trudinger [19] that $f_{\epsilon, S'}^{(3)}(x)$ is thrice continuously differentiable. Letting the radius of S' (and that of S) go to infinity, we have, by p.294 in Kaise and Sheu [22], see also Proposition 3.2 in Ichihara [21], that the $f_{\epsilon, S'}^{(3)}$ converge locally uniformly and in $\mathbb{W}_{\text{loc}}^{1,2}(\mathbb{R}^l)$ to f_ϵ which is a weak solution to $H(x; \lambda, f) = F(\lambda) + \epsilon$. Furthermore, by Lemma 2.4 in Kaise and Sheu [22], the $\mathbb{W}^{1,\infty}(S'')$ -norms of the $f_{\epsilon, S'}^{(3)}$ are uniformly bounded over balls S' for any fixed ball S'' contained in the S' . Therefore, f_ϵ belongs to $\mathbb{W}_{\text{loc}}^{1,\infty}(\mathbb{R}^l)$. By Theorem 6.4 on p.284 in Ladyzhenskaya and Ural'tseva [25], f_ϵ is thrice continuously differentiable.

As in Theorem 4.2 in Kaise and Sheu [22], by using the gradient bound in Lemma 2.4 there (which proof does require f_ϵ to be thrice continuously differentiable), we have that the f_ϵ converge along a subsequence uniformly on compact sets as $\epsilon \rightarrow 0$ to a \mathbb{C}^2 -solution of $H(x; \lambda, f) = F(\lambda)$. That solution, which we denote by f^λ , delivers the infimum in (2.25) and satisfies the Bellman equation, with $\nabla f^\lambda(x)$ obeying the linear growth condition, see Remark 2.5 in Kaise and Sheu [22]. If we assume that $F(\lambda) = -\infty$, then the above reasoning

shows that there exists a solution to $H(x; \lambda, f) = -K$, for all great enough K which leads to a contradiction by the argument of the proof of Theorem 2.6 in Kaise and Sheu [22].

We prove that F is a lower semicontinuous function. Let $\lambda_i \rightarrow \lambda < 1$, as $i \rightarrow \infty$, and let the $F(\lambda_i)$ converge to a finite quantity. By the part just proved, there exist $\tilde{f}_i \in \mathbb{C}^2$ such that $H(x; \lambda_i, \tilde{f}_i) = F(\lambda_i)$, for all x . Furthermore, by a similar reasoning to the one used above the sequence \tilde{f}_i is relatively compact in $L_{\text{loc}}^\infty(\mathbb{R}^l) \cap W_{\text{loc}}^{1,2}(\mathbb{R}^l)$ with limit points being in $W_{\text{loc}}^{1,\infty}(\mathbb{R}^l)$ as well. Subsequential limit \tilde{f} is a \mathbb{C}^2 -function such that $H(x; \lambda, \tilde{f}) = \lim_{i \rightarrow \infty} F(\lambda_i)$. By (2.25), $F(\lambda)$ is the smallest Λ such that there exists \mathbb{C}^2 -function f that satisfies the equation $H(x; \lambda, f) = \Lambda$, for all $x \in \mathbb{R}^l$. Hence, $\lim_{i \rightarrow \infty} F(\lambda_i) \geq F(\lambda)$. The function $F(\lambda)$ is lower semicontinuous at $\lambda = 1$ by definition.

We prove that $F(0) = 0$. Taking $f(x) = 0$ in (2.25) yields $F(0) \leq 0$. Suppose that $F(0) < 0$ and let $f \in \mathbb{C}^2 \cap \mathbb{C}_\ell^1$ be such that, for all $x \in \mathbb{R}^l$,

$$\nabla f(x)^T \theta(x) + \frac{1}{2} |\sigma(x)^T \nabla f(x)|^2 + \frac{1}{2} \text{tr}(\sigma(x) \sigma(x)^T \nabla^2 f(x)) < 0. \quad (3.2)$$

By (2.2), there exists density $m \in \hat{\mathbb{P}}$ such that

$$\int_{\mathbb{R}^l} \left(\nabla h(x)^T \theta(x) + \frac{1}{2} \text{tr}(\sigma(x) \sigma(x)^T \nabla^2 h(x)) \right) m(x) dx = 0, \quad (3.3)$$

for all $h \in \mathbb{C}_0^2$, see, e.g., Corollary 1.4.2 in Bogachev, Krylov, and R  ckner [9]. By (3.2), $\int_{\mathbb{R}^l} \left(\nabla f(x)^T \theta(x) + (1/2) \text{tr}(\sigma(x) \sigma(x)^T \nabla^2 f(x)) \right) m(x) dx$ is well defined, being possibly equal to $-\infty$ and, by monotone convergence,

$$\begin{aligned} & \int_{\mathbb{R}^l} \left(\nabla f(x)^T \theta(x) + \frac{1}{2} \text{tr}(\sigma(x) \sigma(x)^T \nabla^2 f(x)) \right) m(x) dx \\ &= \lim_{R \rightarrow \infty} \int_{x \in \mathbb{R}^l: |x| \leq R} \left(\nabla f(x)^T \theta(x) + \frac{1}{2} \text{tr}(\sigma(x) \sigma(x)^T \nabla^2 f(x)) \right) m(x) dx. \end{aligned}$$

By integration by parts,

$$\begin{aligned} & \int_{x \in \mathbb{R}^l: |x| \leq R} \left(\nabla f(x)^T \theta(x) + \frac{1}{2} \text{tr}(\sigma(x) \sigma(x)^T \nabla^2 f(x)) \right) m(x) dx \\ &= \int_{x \in \mathbb{R}^l: |x| \leq R} \left(\nabla f(x)^T \theta(x) - \frac{1}{2} \nabla f(x)^T \frac{\text{div}(\sigma(x) \sigma(x)^T m(x))}{m(x)} \right) m(x) dx \\ & \quad + \frac{1}{2} \int_{x \in \mathbb{R}^l: |x|=R} \nabla f(x)^T \sigma(x) \sigma(x)^T d(x) m(x) d\tau, \end{aligned}$$

with $d(x)$ denoting the unit outward normal to the sphere $\{x \in \mathbb{R}^l: |x| = R\}$ at point x and with the latter integral being a surface integral. As $\int_{\mathbb{R}^l} |\nabla f(x)| m(x) dx < \infty$,

$$\liminf_{R \rightarrow \infty} \int_{x \in \mathbb{R}^l: |x|=R} |\nabla f(x)^T \sigma(x) \sigma(x)^T d(x)| m(x) d\tau = 0,$$

so letting $R \rightarrow \infty$ appropriately yields the identity

$$\begin{aligned} \int_{\mathbb{R}^l} (\nabla f(x)^T \theta(x) + \frac{1}{2} \operatorname{tr} (\sigma(x) \sigma(x)^T \nabla^2 f(x))) m(x) dx \\ = \int_{\mathbb{R}^l} (\nabla f(x)^T \theta(x) - \frac{1}{2} \nabla f(x)^T \frac{\operatorname{div} (\sigma(x) \sigma(x)^T m(x))}{m(x)}) m(x) dx, \end{aligned} \quad (3.4)$$

implying that the lefthand side is finite. A similar integration by parts in (3.3) yields

$$\int_{\mathbb{R}^l} (\nabla h(x)^T \theta(x) - \frac{1}{2} \nabla h(x)^T \frac{\operatorname{div} (\sigma(x) \sigma(x)^T m(x))}{m(x)}) m(x) dx = 0.$$

Since $m \in \hat{\mathbb{P}}$, this identity extends to $h \in \mathbb{C}^2 \cap \mathbb{C}_\ell^1$, so the righthand side of (3.4) equals zero, which contradicts (3.2). Thus, $F(0) = 0$. \square

Remark 3.1. As a byproduct of the proof, for $\lambda < 1$,

$$\inf_{f \in \mathbb{C}^2} \sup_{x \in \mathbb{R}^l} H(x; \lambda, f) = \inf_{f \in \mathbb{C}^2 \cap \mathbb{C}_\ell^1} \sup_{x \in \mathbb{R}^l} H(x; \lambda, f).$$

Lemma 3.3. *If $\lambda < 1$ and $\mathcal{U}_\lambda \neq \emptyset$, then, for $\nu \in \mathcal{P}$,*

$$\inf_{f \in \mathcal{U}_\lambda} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) = \inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx). \quad (3.5)$$

Proof. Let η be a cut-off function, i.e., a $[0, 1]$ -valued smooth nonincreasing function on \mathbb{R}_+ such that $\eta(y) = 1$ when $y \in [0, 1]$ and $\eta(y) = 0$ when $y \geq 2$. Let us assume, in addition, that the derivative η' does not exceed 2 in absolute value and let $R > 0$. Let $\eta_R(x) = \eta(|x|/R)$. Given $\psi \in \mathbb{C}_0^2$ and $\varphi \in \mathcal{U}_\lambda$, by (2.20) and (2.24),

$$\begin{aligned} H(x; \lambda, \eta_R \psi + (1 - \eta_R) \varphi) &= \frac{1}{2} \nabla \psi(x)^T T_\lambda(x) \nabla \psi(x) \eta_R(x)^2 + S_\lambda(x) \nabla \psi(x) \eta_R(x) \\ &+ \frac{1}{2} \operatorname{tr} (\sigma(x) \sigma(x)^T \nabla^2 \psi(x)) \eta_R(x) + \frac{1}{2} \nabla \varphi(x)^T T_\lambda(x) \nabla \varphi(x) (1 - \eta_R(x))^2 + S_\lambda(x) \nabla \varphi(x) (1 - \eta_R(x)) \\ &+ \frac{1}{2} \operatorname{tr} (\sigma(x) \sigma(x)^T \nabla^2 \varphi(x)) (1 - \eta_R(x)) + \epsilon_R(x) + R_\lambda(x), \end{aligned} \quad (3.6)$$

where

$$\begin{aligned} \epsilon_R(x) &= \frac{1}{2} \nabla \eta_R(x)^T T_\lambda(x) \nabla \eta_R(x) (\psi(x) - \varphi(x))^2 + \nabla \psi(x)^T T_\lambda(x) \nabla \eta_R(x) (\psi(x) - \varphi(x)) \eta_R(x) \\ &+ \nabla \psi(x)^T T_\lambda(x) \nabla \varphi(x) (1 - \eta_R(x)) \eta_R(x) + \nabla \varphi(x)^T T_\lambda(x) \nabla \eta_R(x) (\psi(x) - \varphi(x)) (1 - \eta_R(x)) \\ &+ S_\lambda(x) (\psi(x) - \varphi(x)) \nabla \eta_R(x) + \frac{1}{2} \operatorname{tr} (\sigma(x) \sigma(x)^T ((\psi(x) - \varphi(x)) \nabla^2 \eta_R(x) \\ &\quad + (\nabla \psi(x) - \nabla \varphi(x)) \nabla \eta_R(x)^T)). \end{aligned} \quad (3.7)$$

Replacing on the righthand side of (3.6) $\eta_R(x)^2$ and $(1 - \eta_R(x))^2$ with $\eta_R(x)$ and $1 - \eta_R(x)$, respectively, obtains that

$$H(x; \lambda, \eta_R \psi + (1 - \eta_R) \varphi) \leq \eta_R(x) H(x; \lambda, \psi) + (1 - \eta_R(x)) H(x; \lambda, \varphi) + \epsilon_R(x). \quad (3.8)$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^l} H(x; \lambda, \eta_R \psi + (1 - \eta_R) \varphi) \nu(dx) &\leq \int_{\mathbb{R}^l} \eta_R(x) H(x; \lambda, \psi) \nu(dx) + \sup_{x \in \mathbb{R}^l} (H(x; \lambda, \varphi) \vee 0) \nu(\mathbb{R}^l \setminus B_R) \\ &\quad + \int_{\mathbb{R}^l} \epsilon_R(x) \nu(dx), \end{aligned}$$

where $a \vee b = \max(a, b)$. By dominated convergence, the first integral on the righthand side converges to $\int_{\mathbb{R}^l} H(x; \lambda, \psi) \nu(dx)$, as $R \rightarrow \infty$. Since $|\nabla \eta_R(x)| \leq 4\chi_{\{|x| \geq R\}}(x)/|x|$, $|\nabla \varphi(x)|$ is of, at most, linear growth, by φ being a member of \mathbb{C}_ℓ^1 , so that $\varphi(x)$ grows, at most, quadratically, and since $\int_{\mathbb{R}^l} |x|^2 \nu(dx) < \infty$, by (3.7), one has that

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}^l} \epsilon_R(x) \nu(dx) = 0. \quad (3.9)$$

Since $\psi \eta_R + \varphi(1 - \eta_R) \in \mathcal{U}_\lambda$, agreeing with φ if $|x| > 2R$,

$$\inf_{f \in \mathcal{U}_\lambda} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) \leq \inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx).$$

Conversely, let $\varphi \in \mathcal{U}_\lambda$ and $\psi_R(x) = \eta_R(x) \varphi(x)$. One can see that ψ_R is a \mathbb{C}_0^2 -function. By (2.29), in analogy with (3.8) and (3.9),

$$\int_{\mathbb{R}^l} H(x; \lambda, \psi_R) \nu(dx) \leq \int_{\mathbb{R}^l} (\eta_R(x) H(x; \lambda, \varphi) + (1 - \eta_R(x)) H(x; \lambda, \mathbf{0})) \nu(dx) + \hat{\epsilon}_R,$$

where $\lim_{R \rightarrow \infty} \hat{\epsilon}_R = 0$, with $\mathbf{0}$ representing the function that is equal to zero identically. By Fatou's lemma, $H(x; \lambda, \varphi)$ being bounded from above,

$$\limsup_{R \rightarrow \infty} \int_{\mathbb{R}^l} \eta_R(x) H(x; \lambda, \varphi) \nu(dx) \leq \int_{\mathbb{R}^l} H(x; \lambda, \varphi) \nu(dx). \quad (3.10)$$

By dominated convergence,

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}^l} (1 - \eta_R(x)) H(x; \lambda, \mathbf{0}) \nu(dx) = 0.$$

Hence,

$$\inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) \leq \inf_{f \in \mathcal{U}_\lambda} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx),$$

which concludes the proof of (3.5). □

Remark 3.2. Similarly, it can be shown that, if $\lambda < 1$, then

$$\inf_{f \in \mathbb{C}_b^2} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) = \inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx).$$

(The analogue of (3.10) holds with equality by bounded convergence.)

The following lemma appears in Puhalskii and Stutzer [34].

Lemma 3.4. *If, given $\lambda < 1$, probability measure ν on \mathbb{R}^l is such that the integrals $\int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx)$ are bounded below uniformly over $f \in \mathbb{C}_0^2$, then ν admits density which belongs to $\hat{\mathbb{P}}$.*

Proof. The reasoning follows that of Puhalskii [33], cf. Lemma 6.1, Lemma 6.4, and Theorem 6.1 there. If there exists $\kappa \in \mathbb{R}$ such that $\int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) \geq \kappa$ for all $f \in \mathbb{C}_0^2$, then by (2.24), for arbitrary $\delta > 0$,

$$\delta \int_{\mathbb{R}^l} \frac{1}{2} \operatorname{tr} (\sigma(x) \sigma(x)^T \nabla^2 f(x)) \nu(dx) \geq \kappa - \int_{\mathbb{R}^l} \check{H}(x; \lambda, \delta \nabla f(x)) \nu(dx).$$

On letting

$$\delta = \kappa^{1/2} \left(\int_{\mathbb{R}^l} \nabla f(x)^T T_\lambda(x) \nabla f(x) \nu(dx) \right)^{-1/2},$$

we obtain with the aid of (2.20) and the Cauchy–Schwarz inequality that there exists constant $K_1 > 0$ such that, for all $f \in \mathbb{C}_0^2$,

$$\int_{\mathbb{R}^l} \operatorname{tr} (\sigma(x) \sigma(x)^T \nabla^2 f(x)) \nu(dx) \leq K_1 \left(\int_{\mathbb{R}^l} |\nabla f(x)|^2 \nu(dx) \right)^{1/2}.$$

It follows that the lefthand side extends to a linear functional on $\mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \nu(dx))$, hence, by the Riesz representation theorem, there exists $\nabla h \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \nu(dx))$ such that

$$\int_{\mathbb{R}^l} \operatorname{tr} (\sigma(x) \sigma(x)^T \nabla^2 f(x)) \nu(dx) = \int_{\mathbb{R}^l} \nabla h(x)^T \nabla f(x) \nu(dx) \quad (3.11)$$

and $\int_{\mathbb{R}^l} |\nabla h(x)|^2 \nu(dx) \leq K_1$. Theorem 2.1 in Bogachev, Krylov, and Röckner [8] implies that the measure $\nu(dx)$ has density $m(x)$ with respect to Lebesgue measure which belongs to $L_{\text{loc}}^\xi(\mathbb{R}^l)$ for all $\xi \in (1, l/(l-1))$. It follows that, for arbitrary open ball S in \mathbb{R}^l , there exists $K_2 > 0$ such that for all $f \in \mathbb{C}_0^2$ with support in S ,

$$\left| \int_S \operatorname{tr} (\sigma(x) \sigma(x)^T \nabla^2 f(x)) m(x) dx \right| \leq K_2 \left(\int_S |\nabla f(x)|^{2\xi/(\xi-1)} dx \right)^{(\xi-1)/(2\xi)}.$$

By Theorem 6.1 in Agmon [2], the density m belongs to $\mathbb{W}_{\text{loc}}^{1,\zeta}(S)$ for all $\zeta \in (1, 2l/(2l-1))$. Furthermore, $\nabla h(x) = -\nabla m(x)/m(x)$ so that $\sqrt{m} \in \mathbb{W}^{1,2}(\mathbb{R}^l)$. \square

Remark 3.3. Essentially, (3.11) signifies that one can integrate by parts on the lefthand side, so $m(x)$ needs to be differentiable.

Lemma 3.5. 1. The function $\check{H}(x, \lambda, p)$ is strictly convex in (λ, p) on $(-\infty, 1) \times \mathbb{R}^l$ and is convex on $\mathbb{R} \times \mathbb{R}^l$. The function $H(x; \lambda, f)$ is convex in (λ, f) on $\mathbb{R} \times \mathbb{C}^2$. For $m \in \mathbb{P}$, the function $G(\lambda, f, m)$ is convex in (λ, f) on $\mathbb{R} \times \mathbb{C}_b^2$.

2. Let $m \in \hat{\mathbb{P}}$. Then the function $\check{G}(\lambda, \nabla f, m)$ is convex and lower semicontinuous in $(\lambda, \nabla f)$ on $\mathbb{R} \times \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)$ and is strictly convex on $(-\infty, 1) \times \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)$. If $\lambda < 1$, then the infimum in (2.31) is attained at unique ∇f . If $\lambda = 1$ and the infimum in (2.31) is finite, then it is attained at unique ∇f too. The function $\check{F}(\lambda, m)$ is convex and lower semicontinuous with respect to λ , it is strictly convex on $(-\infty, 1)$, and tends to ∞ superlinearly, as $\lambda \rightarrow -\infty$. If $\lambda < 1$, then

$$\check{F}(\lambda, m) = \inf_{f \in \mathbb{C}^2 \cap \mathbb{C}_\ell^1} \check{G}(\lambda, \nabla f, m) = \inf_{f \in \mathbb{C}_b^2} G(\lambda, f, m). \quad (3.12)$$

If $\lambda < 1$ and $\mathcal{U}_\lambda \neq \emptyset$, then

$$\check{F}(\lambda, m) = \inf_{f \in \mathcal{U}_\lambda} \check{G}(\lambda, \nabla f, m) = \inf_{f \in \mathcal{U}_\lambda} G(\lambda, f, m). \quad (3.13)$$

If $f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)$, then $\check{G}(\lambda, \nabla f, m)$ is differentiable in $\lambda \in (-\infty, 1)$ and

$$\begin{aligned} \frac{d}{d\lambda} \check{G}(\lambda, \nabla f, m) &= \int_{\mathbb{R}^l} (M(u^{\lambda, \nabla f}(x), x) + \lambda |N(u^{\lambda, \nabla f}(x), x)|^2 \\ &\quad + \nabla f(x)^T \sigma(x) N(u^{\lambda, \nabla f}(x), x)) m(x) dx, \end{aligned} \quad (3.14)$$

where $u^{\lambda, \nabla f}(x)$ is defined by (2.17) with $\nabla f(x)$ as p . Furthermore, $\check{F}(\lambda, m)$ is differentiable with respect to λ and

$$\frac{d}{d\lambda} \check{F}(\lambda, m) = \frac{d}{d\lambda} \check{G}(\lambda, \nabla f^{\lambda, m}, m), \quad (3.15)$$

with $\nabla f^{\lambda, m}$ attaining the infimum on the righthand side of (2.31). In addition, if $\check{F}(1, m) < \infty$, then the lefthand derivatives at 1 equal each other as well:

$$\frac{d}{d\lambda} \check{F}(\lambda, m)|_{1-} = \frac{d}{d\lambda} \check{G}(\lambda, \nabla f^{1, m}, m)|_{1-}. \quad (3.16)$$

3. The function $F(\lambda)$ is convex, is continuous for $\lambda < \bar{\lambda}$, and $F(\lambda) \rightarrow \infty$ superlinearly, as $\lambda \rightarrow -\infty$. The functions J_q , J_q^o , and J_q^s are continuous.

Proof. If $\lambda < 1$, then, by (2.16) and (2.18), the Hessian matrix of $\check{H}(x; \lambda, p)$ with respect to

(λ, p) is given by

$$\begin{aligned}\check{H}_{pp}(x; \lambda, p) &= \frac{1}{1-\lambda} \sigma(x) b(x)^T c(x)^{-1} b(x) \sigma(x)^T + \sigma(x) Q_1(x) \sigma(x)^T, \\ \check{H}_{\lambda\lambda}(x; \lambda, p) &= \frac{1}{(1-\lambda)^3} \|a(x) - r(x)\mathbf{1} + b(x)\sigma(x)^T p - b(x)\beta(x)\|_{c(x)^{-1}}^2 + \beta(x)^T Q_1(x) \beta(x), \\ \check{H}_{\lambda p}(x; \lambda, p) &= -\frac{1}{(1-\lambda)^2} (a(x) - r(x)\mathbf{1} + b(x)\sigma(x)^T p - b(x)\beta(x))^T c(x)^{-1} b(x) \sigma(x)^T \\ &\quad + \beta(x)^T Q_1(x) \sigma(x)^T.\end{aligned}$$

We show that it is positive definite. More specifically, we prove that for all $\tau \in \mathbb{R}$ and $y \in \mathbb{R}^l$ such that $\tau^2 + |y|^2 \neq 0$,

$$\tau^2 \check{H}_{\lambda\lambda}(x; \lambda, p) + y^T \check{H}_{pp}(x; \lambda, p) y + 2\tau \check{H}_{\lambda p}(x; \lambda, p) y > 0.$$

Since $\check{H}_{pp}(x; \lambda, p)$ is a positive definite matrix by condition (N), the latter inequality holds when $\tau = 0$. Assuming $\tau \neq 0$, we need to show that

$$\check{H}_{\lambda\lambda}(x; \lambda, p) + y^T \check{H}_{pp}(x; \lambda, p) y + 2\check{H}_{\lambda p}(x; \lambda, p) y > 0. \quad (3.17)$$

Let, for $d_1 = (v_1(x), w_1(x))$ and $d_2 = (v_2(x), w_2(x))$, where $v_1(x) \in \mathbb{R}^n$, $w_1(x) \in \mathbb{R}^k$, $v_2(x) \in \mathbb{R}^n$, $w_2(x) \in \mathbb{R}^k$, and $x \in \mathbb{R}^l$, the inner product be defined by $d_1 \cdot d_2 = v_1(x)^T c(x)^{-1} v_2(x) + w_1(x)^T w_2(x)$. By the Cauchy-Schwarz inequality, applied to $d_1 = ((1-\lambda)^{-3/2}(a(x) - r(x)\mathbf{1} + b(x)\sigma(x)^T p - b(x)\beta(x)), Q_1(x)\beta(x))$ and $d_2 = ((1-\lambda)^{-1/2}b(x)\sigma(x)^T y, Q_1(x)\sigma(x)^T y)$, we have that $(\check{H}_{\lambda p}(x; \lambda, p)y)^2 < y^T \check{H}_{pp}(x; \lambda, p) y \check{H}_{\lambda\lambda}(x; \lambda, p)$, with the inequality being strict because, by part 2 of condition (N), $Q_1(x)\beta(x)$ is not a scalar multiple of $Q_1(x)\sigma(x)^T y$. Thus, (3.17) holds, so the function $\check{H}(x; \lambda, p)$ is strictly convex in (λ, p) on $(-\infty, 1) \times \mathbb{R}^l$, for all $x \in \mathbb{R}^l$.

Since by (2.16) and (2.18), $\check{H}(x; \lambda_n, p_n) \rightarrow \check{H}(x; 1, p) \leq \infty$ as $\lambda_n \uparrow 1$ and $p_n \rightarrow p$, and $\check{H}(x; \lambda, p) = \infty$ if $\lambda > 1$, the function $\check{H}(x; \lambda, p)$ is convex in (λ, p) on $\mathbb{R} \times \mathbb{R}^l$. By (2.24), the function $H(x; \lambda, f)$ is convex in (λ, f) on $\mathbb{R} \times \mathbb{C}^2$. By (2.28), for any $m \in \mathbb{P}$, $G(\lambda, f, m)$ is convex in (λ, f) on $\mathbb{R} \times \mathbb{C}_b^2$.

Let $m \in \hat{\mathbb{P}}$. By (2.30) and the strict convexity of \check{H} , $\check{G}(\lambda, \nabla f, m)$ is strictly convex in $(\lambda, \nabla f) \in (-\infty, 1) \times \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)$. Let us note that, by (2.18), for $\epsilon > 0$,

$$\begin{aligned}\check{H}(x; \lambda, p) &\geq -\frac{1}{2} \|a(x) - r(x)\mathbf{1} - \lambda b(x)\beta(x) + b(x)\sigma(x)^T p\|_{c(x)^{-1}}^2 + \frac{1}{2} \lambda^2 |\beta(x)|^2 \\ &\quad + \lambda(r(x) - \alpha(x) + \frac{1}{2} |\beta(x)|^2 - \beta(x)^T \sigma(x)^T p) + p^T \theta(x) + \frac{1}{2} |\sigma(x)^T p|^2 \\ &\geq -\frac{1}{2} \left((1+\epsilon) \|b(x)\sigma(x)^T p\|_{c(x)^{-1}}^2 + \left(1 + \frac{1}{\epsilon}\right) \|a(x) - r(x)\mathbf{1} - \lambda b(x)\beta(x)\|_{c(x)^{-1}}^2 \right) \\ &\quad + \frac{1}{2} \lambda^2 |\beta(x)|^2 + \lambda(r(x) - \alpha(x) + \frac{1}{2} |\beta(x)|^2) + p^T (\theta(x) - \lambda \sigma(x) \beta(x)) + \frac{1}{2} |\sigma(x)^T p|^2 \\ &= \frac{1}{2} \|p\|_{Q_{1,\epsilon}(x)}^2 + \frac{1}{2} \left(1 + \frac{1}{\epsilon}\right) \|a(x) - r(x)\mathbf{1} - \lambda b(x)\beta(x)\|_{c(x)^{-1}}^2 \\ &\quad + \frac{1}{2} \lambda^2 |\beta(x)|^2 + \lambda(r(x) - \alpha(x) + \frac{1}{2} |\beta(x)|^2) + p^T (\theta(x) - \lambda \sigma(x) \beta(x)), \quad (3.18)\end{aligned}$$

where $Q_{1,\epsilon}(x) = Q_1(x) - \epsilon\sigma(x)b(x)^T c(x)^{-1}b(x)\sigma(x)^T$. Since $Q_1(x)$ is uniformly positive definite, so is $Q_{1,\epsilon}(x)$, provided ϵ is small enough. By (3.18), (2.30), and by the facts that $\int_{\mathbb{R}^l} |x|^2 m(x) dx < \infty$ and $\int_{\mathbb{R}^l} |\nabla m(x)|^2 / m(x) dx < \infty$, $\check{G}(\lambda, \nabla f, m)$ tends to infinity as the $\mathbb{L}^2(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)$ -norm of ∇f tends to infinity, locally uniformly over λ . Since, in addition, $\check{G}(\lambda, \nabla f, m)$ is strictly convex in $(\lambda, \nabla f)$, the infimum on the right-hand side of (2.31) is attained at unique ∇f , if finite, see, e.g., Proposition 1.2 on p.35 in Ekeland and Temam [14]. (If $\lambda < 1$, then $\check{G}(\lambda, \nabla f, m) < \infty$, for all $\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)$, by (2.20) and (2.30).) Hence, the righthand side of (2.31) is strictly convex in λ on $(-\infty, 1)$. (For, let $\inf_{\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)} \check{G}(\lambda_i, \nabla f, m) = \check{G}(\lambda_i, \nabla f_i, m)$, for $i = 1, 2$. Then $\inf_{\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)} \check{G}((\lambda_1 + \lambda_2)/2, \nabla f, m) \leq \check{G}((\lambda_1 + \lambda_2)/2, (\nabla f_1 + \nabla f_2)/2, m) < (\check{G}(\lambda_1, \nabla f_1, m) + \check{G}(\lambda_2, \nabla f_2, m))/2 = (\inf_{\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)} \check{G}(\lambda_1, \nabla f, m) + \inf_{\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)} \check{G}(\lambda_2, \nabla f, m))/2$.)

By (3.18), by $\check{H}(x; \lambda, p)$ being a lower semicontinuous function of (λ, p) with values in $\mathbb{R} \cup \{+\infty\}$, by (2.30) and Fatou's lemma, $\check{G}(\lambda, \nabla f, m)$ is lower semicontinuous in $(\lambda, \nabla f)$ on $\mathbb{R} \times \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)$. By a similar argument to that in Proposition 1.7 on p.14 in Aubin [3] or Proposition 5 on p.12 in Aubin and Ekeland [4], the function $\check{F}(\lambda, m)$ is lower semicontinuous in λ . More specifically, let $\lambda_i \rightarrow \lambda$ and let $K_1 = \liminf_{i \rightarrow \infty} \check{F}(\lambda_i, m)$. Assuming that $K_1 < \infty$, by (2.31), for all i great enough,

$$\check{F}(\lambda_i, m) = \inf_{\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx): \check{G}(\lambda_i, \nabla f, m) \leq K_1 + 1} \check{G}(\lambda_i, \nabla f, m).$$

By (2.30) and (3.18), there exists K_2 such that, for all i , if $\check{G}(\lambda_i, \nabla f, m) \leq K_1 + 1$, then $\int_{\mathbb{R}^l} |\nabla f(x)|^2 m(x) dx \leq K_2$. The set of the latter ∇f being weakly compact in $\mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)$ and the function $\check{G}(\lambda, \nabla f, m)$ being convex and lower semicontinuous in ∇f , there exist ∇f_i such that $\check{F}(\lambda_i, m) = \check{G}(\lambda_i, \nabla f_i, m)$. Extracting a suitable subsequence of ∇f_i that weakly converges to some ∇f and invoking the lower semicontinuity of $\check{G}(\lambda, \nabla f, m)$ in $(\lambda, \nabla f)$ yields

$$\begin{aligned} \liminf_{i \rightarrow \infty} \check{F}(\lambda_i, m) &= \liminf_{i \rightarrow \infty} \inf_{\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx): \check{G}(\lambda_i, \nabla f, m) \leq K_1 + 1} \check{G}(\lambda_i, \nabla f, m) \\ &\geq \liminf_{i \rightarrow \infty} \inf_{\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx): \int_{\mathbb{R}^l} |\nabla f(x)|^2 m(x) dx \leq K_2} \check{G}(\lambda_i, \nabla f, m) \\ &= \liminf_{i \rightarrow \infty} \check{G}(\lambda_i, \nabla f_i, m) \geq \check{G}(\lambda, \nabla f, m) \geq \check{F}(\lambda, m). \end{aligned}$$

We have proved that the function $\check{F}(\lambda, m)$ is lower semicontinuous in λ . It follows that the function $\sup_{m \in \mathbb{P}} \check{F}(\lambda, m)$ is lower semicontinuous.

Let us show that the gradients of functions from $\mathbb{C}^2 \cap \mathbb{C}_\ell^1$ make up a dense subset of $\mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \hat{m}(x) dx)$. Let $f \in \mathbb{C}_\ell^1$ and let $\eta(y)$ represent a cut-off function, i.e., a $[0, 1]$ -valued smooth nonincreasing function on \mathbb{R}_+ such that $\eta(y) = 1$ when $y \in [0, 1]$ and $\eta(y) = 0$

when $y \geq 2$. Let $R > 0$. The function $f(x)\eta(|x|/R)$ belongs to \mathbb{C}_0^1 . In addition,

$$\begin{aligned} \int_{\mathbb{R}^l} |\nabla f(x) - \nabla(f(x)\eta(\frac{|x|}{R}))|^2 m(x) dx &\leq 2 \int_{\mathbb{R}^l} |\nabla f(x)|^2 (1 - \eta(\frac{|x|}{R}))^2 m(x) dx \\ &\quad + \frac{2}{R^2} \int_{\mathbb{R}^l} f(x)^2 \eta'(\frac{|x|}{R})^2 m(x) dx, \end{aligned}$$

where η' stands for the derivative of η . Since $\int_{\mathbb{R}^l} |x|^2 m(x) dx$ converges, the righthand side of the latter inequality tends to 0 as $R \rightarrow \infty$. Hence, $\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \hat{m}(x) dx)$. On the other hand, the gradients of \mathbb{C}_0^1 -functions can be approximated with the gradients of $\mathbb{C}^2 \cap \mathbb{C}_\ell^1$ -functions in $\mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \hat{m}(x) dx)$, which ends the proof.

On recalling (2.31), we obtain the leftmost equality in (3.12). Similarly, since $G(\lambda, f, m) = \check{G}(\lambda, \nabla f, m)$ when $f \in \mathbb{C}_0^2$ and the gradients of \mathbb{C}_0^2 -functions are dense in $\mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)$, the rightmost side of (3.12) equals the leftmost side. For (3.13), we recall Lemma 3.3 and note that, as the proof of Lemma 3.2 shows, $G(\lambda, f, m) = \check{G}(\lambda, \nabla f, m)$ when $f \in \mathcal{U}_\lambda$ and $\lambda < 1$.

By (2.20) and (2.15), as $\lambda \rightarrow -\infty$,

$$\lim_{\lambda \rightarrow -\infty} \frac{1}{\lambda^2} \inf_{p \in \mathbb{R}^l} (\check{H}(x; \lambda, p) - \frac{1}{2} p^T \sigma(x) \sigma(x)^T \frac{\nabla m(x)}{m(x)}) = \frac{1}{2} \|\beta(x)\|_{Q_2(x)}^2.$$

The latter quantity being positive by the second part of condition (N) implies, by (2.31), that $\liminf_{\lambda \rightarrow -\infty} (1/\lambda^2) \check{F}(\lambda, m) > 0$, so, $\liminf_{\lambda \rightarrow -\infty} (1/\lambda^2) \inf_{f \in \mathbb{C}_0^2} G(\lambda, f, m) > 0$. By (2.25), (2.27), and (2.28), $F(\lambda) \geq \inf_{f \in \mathbb{C}_0^2} G(\lambda, f, m)$, so, $\liminf_{\lambda \rightarrow -\infty} F(\lambda)/\lambda^2 > 0$. Therefore, for all q from a bounded set, the supremum in (2.39a) can be taken over λ from the same compact set, which implies that J_q is continuous. With J_q^o and J_q^s , a similar reasoning applies. Since $\sup_{x \in \mathbb{R}^l} H(x; \lambda, f)$ is a convex function of (λ, f) , by (2.25), $F(\lambda)$ is convex. Being finite, it is continuous for $\lambda < \bar{\lambda}$.

We prove the differentiability properties. The assertion in (3.14) follows by Theorem 4.13 on p.273 in Bonnans and Shapiro [10] and dominated convergence, once we recall (2.20) and (2.30). Equation (3.15) is obtained similarly, with $\check{G}(\cdot, \cdot, m)$ as $f(\cdot, \cdot)$, with λ as u , and with ∇f as x , respectively, in the hypotheses of Theorem 4.13 on p.273 in Bonnans and Shapiro [10]. In some more detail, $\check{G}(\lambda, \nabla f, m)$ and $d\check{G}(\lambda, \nabla f, m)/d\lambda$ are continuous functions of $(\lambda, \nabla f)$ by (2.16), (2.17), and (2.30). The inf-compactness condition on p.272 in Bonnans and Shapiro [10] holds because, as it has been shown in the proof of the lower semicontinuity of $\check{F}(\lambda, m)$, the infimum on the righthand side of (2.31) can be taken over the same weakly compact subset of $\mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)$ for all λ from a compact subset of $(-\infty, 1)$. For (3.16), one can also apply the reasoning of the proof of Theorem 4.13 on p.273 in Bonnans and Shapiro [10]. Although the hypotheses of the theorem are not satisfied, the proof on pp.274,275 goes through, the key being that the function $\check{G}(\lambda, \nabla f, m)$ tends to infinity uniformly over λ close enough to 1 on the left, as the $\mathbb{L}^2(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)$ -norm of ∇f tends to infinity. □

Remark 3.4. If condition (N) is not assumed, then strict convexity in the statement has to be replaced with convexity.

Remark 3.5. If $\beta(x) = 0$, then $F(\lambda)/\lambda^2$ tends to zero as $\lambda \rightarrow -\infty$. Furthermore,

$$\liminf_{\lambda \rightarrow -\infty} \frac{1}{|\lambda|} \inf_{f \in \mathbb{C}_0^2} G(\lambda, f, m) \geq - \int_{\mathbb{R}^l} r(x) m(x) dx,$$

so that

$$\liminf_{\lambda \rightarrow -\infty} \frac{F(\lambda)}{|\lambda|} \geq - \inf_{x \in \mathbb{R}^l} r(x).$$

Consequently, if $\inf_{x \in \mathbb{R}^l} r(x) < q$, then $\lambda q - F(\lambda)$ tends to $-\infty$ as $\lambda \rightarrow -\infty$, so $\sup_{\lambda \in \mathbb{R}} (\lambda q - F(\lambda))$ is attained. That might not be the case if $\inf_{x \in \mathbb{R}^l} r(x) \geq q$. For instance, if the functions $a(x)$, $r(x)$, $b(x)$, and $\sigma(x)$ are constant and q is small enough, then the derivative of $\lambda q - F(\lambda)$ is positive for all $\lambda < 0$. In particular, J_q , J_q^s , or J_q^o might not be continuous at $\inf_{x \in \mathbb{R}^l} r(x)$, J_q^s being rightcontinuous and J_q^o being leftcontinuous regardless.

Lemma 3.6. 1. The function $\lambda q - \check{F}(\lambda, m)$ has saddle point $(\hat{\lambda}, \hat{m})$ in $(-\infty, \bar{\lambda}] \times \hat{\mathbb{P}}$, with $\hat{\lambda}$ being specified uniquely. In addition, $\hat{\lambda} q - F(\hat{\lambda}) = \sup_{\lambda \in \mathbb{R}} (\lambda q - F(\lambda))$. If $\lambda \leq \bar{\lambda}$, then $F(\lambda) = \sup_{m \in \hat{\mathbb{P}}} \check{F}(\lambda, m)$.

2. Suppose that $\hat{\lambda} < 1$. Then the function $\lambda q - \check{G}(\lambda, \nabla f, m)$, being concave in (λ, f) and convex in m , has saddle point $(\hat{\lambda}, \hat{f}, \hat{m})$ in $(-\infty, \bar{\lambda}] \times (\mathbb{C}^2 \cap \mathbb{C}_\ell^1) \times \hat{\mathbb{P}}$, with $\nabla \hat{f}$ and \hat{m} being specified uniquely. Equations (2.32) and (2.33) hold.

3. Suppose that $\hat{\lambda} = 1$. Then there exists unique $\nabla \hat{f} \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \hat{m}(x) dx)$ such that $\check{F}(1, \hat{m}) = \check{G}(1, \nabla \hat{f}, \hat{m})$, $a(x) - r(x)\mathbf{1} - b(x)\beta(x) + b(x)\sigma(x)^T \nabla \hat{f}(x) = 0$ $\hat{m}(x) dx$ -a.e. and

$$\int_{\mathbb{R}^l} (\nabla h(x)^T (-\sigma(x)\beta(x) + \theta(x) + \sigma(x)\sigma(x)^T \nabla \hat{f}(x)) + \frac{1}{2} \text{tr}(\sigma(x)\sigma(x)^T \nabla^2 h(x))) \hat{m}(x) dx = 0,$$

for all $h \in \mathbb{C}_0^2$ such that $b(x)\sigma(x)^T \nabla h(x) = 0$ $\hat{m}(x) dx$ -a.e.

Proof. Let $\mathcal{U} = \{(\lambda, f) : f \in \mathcal{U}_\lambda\}$. It is a convex set by $H(x; \lambda, f)$ being convex in (λ, f) . Let $\tilde{q} \in \mathbb{R}$. When $(\lambda, f) \in \mathcal{U}$ and $\nu \in \mathcal{P}$, the function $\lambda \tilde{q} - \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx)$ is well defined, being possibly equal to $+\infty$, is concave in (λ, f) , is convex and lower semicontinuous in ν , and is inf-compact in ν , provided $\lambda < 0$, the latter property holding by Lemma 3.1. Theorem 7 on p.319 in Aubin and Ekeland [4], whose proof applies to the case of the function $f(x, y)$ in the statement of the theorem taking values in $\mathbb{R} \cup \{+\infty\}$ yields the identity

$$\inf_{\nu \in \mathcal{P}} \sup_{(\lambda, f) \in \mathcal{U}} (\lambda \tilde{q} - \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx)) = \sup_{(\lambda, f) \in \mathcal{U}} \inf_{\nu \in \mathcal{P}} (\lambda \tilde{q} - \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx)), \quad (3.19)$$

with the infimum on the lefthand side being attained, at $\hat{\nu}$. If ν has no density with respect to Lebesgue measure that belongs to $\hat{\mathbb{P}}$, then, by Lemma 3.4, the supremum on the lefthand side equals $+\infty$. Hence, the infimum on the lefthand side may be taken over ν with densities

from $\hat{\mathbb{P}}$, in particular, it may be assumed that $\hat{\nu}(dx) = \hat{m}(x) dx$, where $\hat{m} \in \hat{\mathbb{P}}$. We thus have that

$$\inf_{m \in \hat{\mathbb{P}}} \sup_{\lambda \in \mathbb{R}} (\lambda \tilde{q} - \inf_{f \in \mathcal{U}_\lambda} G(\lambda, f, m)) = \sup_{\lambda \in \mathbb{R}} (\lambda \tilde{q} - \inf_{f \in \mathbb{C}^2 \cap \mathbb{C}_\ell^1} \sup_{x \in \mathbb{R}^l} H(x; \lambda, f)). \quad (3.20)$$

(We recall that if $\mathcal{U}_\lambda = \emptyset$ then $\inf_{f \in \mathcal{U}_\lambda} = \infty$.) By part 2 of Lemma 3.5, $\inf_{f \in \mathcal{U}_\lambda} G(\lambda, f, m) \rightarrow \infty$ superlinearly, as $\lambda \rightarrow -\infty$, which, when combined with (3.18), implies that both sides of (3.20) are finite. We have that

$$\inf_{m \in \hat{\mathbb{P}}} \sup_{\lambda \in \mathbb{R}} (\lambda \tilde{q} - \inf_{f \in \mathcal{U}_\lambda} G(\lambda, f, m)) \geq \sup_{\lambda \in \mathbb{R}} \inf_{m \in \hat{\mathbb{P}}} (\lambda \tilde{q} - \inf_{f \in \mathcal{U}_\lambda} G(\lambda, f, m)) \geq \sup_{\lambda \in \mathbb{R}} (\lambda \tilde{q} - \inf_{f \in \mathcal{U}_\lambda} \sup_{m \in \hat{\mathbb{P}}} G(\lambda, f, m)).$$

The latter rightmost side being equal to the rightmost side of (3.20) and the definition of $F(\lambda)$ in (2.25) imply that

$$\sup_{\lambda \in \mathbb{R}} (\lambda \tilde{q} - \sup_{m \in \hat{\mathbb{P}}} \inf_{f \in \mathcal{U}_\lambda} G(\lambda, f, m)) = \sup_{\lambda \in \mathbb{R}} (\lambda \tilde{q} - \inf_{f \in \mathbb{C}^2 \cap \mathbb{C}_\ell^1} \sup_{x \in \mathbb{R}^l} H(x; \lambda, f)) = \sup_{\lambda \in \mathbb{R}} (\lambda \tilde{q} - F(\lambda)). \quad (3.21)$$

Therefore, for arbitrary $\lambda \in \mathbb{R}$ and $\tilde{q} \in \mathbb{R}$,

$$\sup_{m \in \hat{\mathbb{P}}} \inf_{f \in \mathcal{U}_\lambda} G(\lambda, f, m) \geq \lambda \tilde{q} - \sup_{\tilde{\lambda} \in \mathbb{R}} (\tilde{\lambda} \tilde{q} - F(\tilde{\lambda})). \quad (3.22)$$

Since F is a lower semicontinuous and convex function, it equals its bidual, so, taking supremum over \tilde{q} in (3.22) yields the inequality $\sup_{m \in \hat{\mathbb{P}}} \inf_{f \in \mathcal{U}_\lambda} G(\lambda, f, m) \geq F(\lambda)$. The opposite inequality being true by the definition of $F(\lambda)$ (see (2.25)) implies that

$$F(\lambda) = \sup_{m \in \hat{\mathbb{P}}} \inf_{f \in \mathcal{U}_\lambda} G(\lambda, f, m). \quad (3.23)$$

In addition, owing to Lemma 3.5, if $\lambda < \bar{\lambda}$, then

$$F(\lambda) = \sup_{m \in \hat{\mathbb{P}}} \inf_{f \in \mathbb{C}^2 \cap \mathbb{C}_\ell^1} \check{G}(\lambda, \nabla f, m) = \sup_{m \in \hat{\mathbb{P}}} \check{F}(\lambda, m). \quad (3.24)$$

By convexity and lower semicontinuity, the latter equality extends to $\lambda = \bar{\lambda}$.

Since the infimum on the lefthand side of (3.20) is attained at \hat{m} , by (3.23),

$$\begin{aligned} \sup_{\lambda \in \mathbb{R}} (\lambda q - \inf_{f \in \mathcal{U}_\lambda} G(\lambda, f, \hat{m})) &= \inf_{m \in \hat{\mathbb{P}}} \sup_{\lambda \in \mathbb{R}} (\lambda q - \inf_{f \in \mathcal{U}_\lambda} G(\lambda, f, m)) \\ &= \sup_{\lambda \in \mathbb{R}} \inf_{m \in \hat{\mathbb{P}}} (\lambda q - \inf_{f \in \mathcal{U}_\lambda} G(\lambda, f, m)). \end{aligned} \quad (3.25)$$

By convexity of $\inf_{f \in \mathcal{U}_\lambda} G(\lambda, f, \hat{m})$ and of $\check{F}(\lambda, \hat{m})$ in λ , we have that $\inf_{f \in \mathcal{U}_{\bar{\lambda}}} G(\bar{\lambda}, f, \hat{m})$ and $\check{F}(\bar{\lambda}, \hat{m})$ are greater than or equal to their respective lefthand limits at $\bar{\lambda}$, so, by the fact that $\mathcal{U}_\lambda = \emptyset$ if $\lambda > \bar{\lambda}$ and part 2 of Lemma 3.5,

$$\sup_{\lambda \in \mathbb{R}} (\lambda q - \inf_{f \in \mathcal{U}_\lambda} G(\lambda, f, \hat{m})) = \sup_{\lambda < \bar{\lambda}} (\lambda q - \inf_{f \in \mathcal{U}_\lambda} G(\lambda, f, \hat{m})) = \sup_{\lambda < \bar{\lambda}} (\lambda q - \check{F}(\lambda, \hat{m})) = \sup_{\lambda \leq \bar{\lambda}} (\lambda q - \check{F}(\lambda, \hat{m})).$$

Similarly,

$$\inf_{m \in \hat{\mathbb{P}}} \sup_{\lambda \in \mathbb{R}} (\lambda q - \inf_{f \in \mathcal{U}_\lambda} G(\lambda, f, m)) = \inf_{m \in \hat{\mathbb{P}}} \sup_{\lambda \leq \bar{\lambda}} (\lambda q - \check{F}(\lambda, m))$$

and

$$\sup_{\lambda \in \mathbb{R}} \inf_{m \in \hat{\mathbb{P}}} (\lambda q - \inf_{f \in \mathcal{U}_\lambda} G(\lambda, f, m)) = \sup_{\lambda \leq \bar{\lambda}} \inf_{m \in \hat{\mathbb{P}}} (\lambda q - \check{F}(\lambda, m)),$$

so, by (3.25),

$$\sup_{\lambda \leq \bar{\lambda}} (\lambda q - \check{F}(\lambda, \hat{m})) = \inf_{m \in \hat{\mathbb{P}}} \sup_{\lambda \leq \bar{\lambda}} (\lambda q - \check{F}(\lambda, m)) = \sup_{\lambda \leq \bar{\lambda}} \inf_{m \in \hat{\mathbb{P}}} (\lambda q - \check{F}(\lambda, m)).$$

Since, by Lemma 3.5, $\check{F}(\lambda, \hat{m})$ is a lower semicontinuous function of λ and $\check{F}(\lambda, \hat{m}) \rightarrow \infty$ superlinearly as $\lambda \rightarrow -\infty$, the supremum on the leftmost side is attained at some $\hat{\lambda}$. It follows that $(\hat{\lambda}, \hat{m})$ is a saddle point of $\lambda q - \check{F}(\lambda, m)$ in $(-\infty, \bar{\lambda}] \times \hat{\mathbb{P}}$. By Lemma 3.5, $\lambda q - \check{F}(\lambda, m)$ is a strictly concave function of λ on $(-\infty, 1)$ for all m , so $\hat{\lambda}$ is specified uniquely, see Proposition 1.5 on p.169 in Ekeland and Temam [14].

We obtain that

$$\begin{aligned} \sup_{\lambda \in \mathbb{R}} (\lambda q - F(\lambda)) &= \sup_{\lambda \leq \bar{\lambda}} (\lambda q - F(\lambda)) = \sup_{\lambda \leq \bar{\lambda}} (\lambda q - \sup_{m \in \hat{\mathbb{P}}} \check{F}(\lambda, m)) = \hat{\lambda} q - \check{F}(\hat{\lambda}, \hat{m}) \\ &= \hat{\lambda} q - \sup_{m \in \hat{\mathbb{P}}} \check{F}(\hat{\lambda}, m) = \hat{\lambda} q - F(\hat{\lambda}). \end{aligned}$$

Part 1 has been proved.

Suppose that $\hat{\lambda} < 1$ and let $\hat{f} = f^{\hat{\lambda}}$, where f^λ is introduced in Lemma 3.2. Since $H(x; \hat{\lambda}, \hat{f}) = F(\hat{\lambda})$ for all $x \in \mathbb{R}^l$, we have that $F(\hat{\lambda}) = G(\hat{\lambda}, \hat{f}, m) = \check{G}(\hat{\lambda}, \nabla \hat{f}, m)$, for all $m \in \hat{\mathbb{P}}$. By (2.29),

$$\inf_{f \in \mathbb{C}^2 \cap \mathbb{C}_\ell^1} \sup_{m \in \hat{\mathbb{P}}} \check{G}(\hat{\lambda}, \nabla f, m) \leq \sup_{m \in \hat{\mathbb{P}}} \check{G}(\hat{\lambda}, \nabla \hat{f}, m) = F(\hat{\lambda}) = \check{G}(\hat{\lambda}, \nabla \hat{f}, \hat{m}). \quad (3.26)$$

By (3.24), the inequality is actually equality and (\hat{f}, \hat{m}) is a saddle point of $\check{G}(\hat{\lambda}, \nabla f, m)$ in $(\mathbb{C}^2 \cap \mathbb{C}_\ell^1) \times \hat{\mathbb{P}}$, see, e.g., Proposition 2.156 on p.104 in Bonnans and Shapiro [10] or Proposition 1.2 on p.167 in Ekeland and Temam [14]. As a result,

$$\inf_{\tilde{f} \in \mathbb{C}^2 \cap \mathbb{C}_\ell^1} \check{G}(\hat{\lambda}, \nabla \tilde{f}, \hat{m}) = \check{G}(\hat{\lambda}, \nabla \hat{f}, \hat{m}). \quad (3.27)$$

By (2.31) and $\mathbb{C}^2 \cap \mathbb{C}_\ell^1$ being dense in $\mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \hat{m}(x) dx)$, the lefthand side of (3.27) equals $\check{F}(\hat{\lambda}, \hat{m})$, so, the infimum on the righthand side of (2.31) for $m = \hat{m}$ is attained at the gradient of the $\mathbb{C}^2 \cap \mathbb{C}_\ell^1$ -function \hat{f} .

The following reasoning shows that $(\hat{\lambda}, \hat{f}, \hat{m})$ is a saddle point of $\lambda q - \check{G}(\lambda, \nabla f, m)$ in $(-\infty, \bar{\lambda}] \times (\mathbb{C}^2 \cap \mathbb{C}_\ell^1) \times \hat{\mathbb{P}}$. Let $\lambda \leq \bar{\lambda}$, $f \in \mathbb{C}^2 \cap \mathbb{C}_\ell^1$, and $m \in \hat{\mathbb{P}}$. Since $\check{G}(\hat{\lambda}, \nabla \hat{f}, \hat{m}) \geq \check{G}(\hat{\lambda}, \nabla \hat{f}, m)$ by (\hat{f}, \hat{m}) being a saddle point of $\check{G}(\hat{\lambda}, \nabla f, m)$, we have that

$$\hat{\lambda} q - \check{G}(\hat{\lambda}, \nabla \hat{f}, \hat{m}) \leq \hat{\lambda} q - \check{G}(\hat{\lambda}, \nabla \hat{f}, m). \quad (3.28)$$

By (3.27), by (2.31), and by $(\hat{\lambda}, \hat{m})$ being a saddle point of $\lambda q - \check{F}(\lambda, m)$,

$$\hat{\lambda}q - \check{G}(\hat{\lambda}, \nabla \hat{f}, \hat{m}) = \hat{\lambda}q - \check{F}(\hat{\lambda}, \hat{m}) \geq \lambda q - \hat{F}(\lambda, \hat{m}) \geq \lambda q - \check{G}(\lambda, \nabla f, \hat{m}). \quad (3.29)$$

Putting together (3.28) and (3.29) yields the required property.

Since $(\hat{\lambda}, \hat{f}, \hat{m})$ is a saddle point of $\lambda q - \check{G}(\lambda, \nabla f, m)$ in $(-\infty, \bar{\lambda}] \times (\mathbb{C}^2 \cap \mathbb{C}_\ell^1) \times \hat{\mathbb{P}}$ and $\lambda q - \check{G}(\lambda, \nabla f, m)$ is strictly concave in $(\lambda, \nabla f)$ for all m , the pair $(\hat{\lambda}, \nabla \hat{f})$ is specified uniquely, see Proposition 1.5 on p.169 of Ekeland and Temam [14]. Equation (2.32) follows by Lemma 3.2. Since \hat{f} is a stationary point of $\check{G}(\hat{\lambda}, \nabla f, \hat{m})$, the directional derivatives of $\check{G}(\hat{\lambda}, \nabla f, \hat{m})$ at \hat{f} are equal to zero, cf. Proposition 1.6 on p.169 in Ekeland and Temam [14]. By (2.30),

$$\int_{\mathbb{R}^l} \left(\check{H}_p(x; \hat{\lambda}, \nabla \hat{f}(x)) - \frac{1}{2} \frac{(\operatorname{div}(\sigma(x)\sigma(x)^T \hat{m}(x)))^T}{\hat{m}(x)} \right) \nabla h(x) \hat{m}(x) dx = 0, \quad (3.30)$$

for all $h \in \mathbb{C}_0^2$. Integration by parts yields (2.33). In more detail, by Theorem 4.17 on p.276 in Bonnans and Shapiro [10], if $\lambda < 1$, then the function $\sup_{u \in \mathbb{R}^n} (M(u, x) + \lambda |N(u, x)|^2/2 + p^T \sigma(x)N(u, x))$, with the supremum being attained at unique point $\tilde{u}(x)$, has a derivative with respect to p given by $(\sigma(x)N(\tilde{u}(x), x))^T$, which, when combined with (2.28) and (3.30), yields (2.33). By Example 1.7.11 (or Example 1.7.14) in Bogachev, Krylov, and Röckner [9], \hat{m} is specified uniquely by (2.33). Part 2 has been proved.

If $\hat{\lambda} = 1$, then $\check{F}(1, \hat{m}) < \infty$. By Lemma 3.5, $\nabla \hat{f}$ exists. The other properties in part 3 follow by (2.22) and (2.23). \square

Remark 3.6. If $\hat{\lambda} < 0$, then $H(x; \hat{\lambda}, f_\kappa) \rightarrow -\infty$ as $|x| \rightarrow \infty$, where $\kappa > 0$ and is small enough, see Puhalskii and Stutzer [34]. In that case, the theory in Keise and Sheu [22] and Ichihara [21] yields an alternative approach to the existence of solution \hat{m} to (2.32). If $\hat{\lambda} > 0$, however, those results do not seem to apply.

Remark 3.7. If the suprema in (3.24) were attained, then $F(\lambda)$ would be strictly convex.

Lemma 3.7. *Suppose that $\hat{\lambda} \leq 0$. Then, for $\kappa > 0$ small enough,*

$$\inf_{f \in \mathcal{A}_\kappa} \sup_{\nu \in \mathcal{P}} \int_{\mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho) \nu(dx) = \sup_{\nu \in \mathcal{P}} \inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} \check{H}(x; \hat{\lambda}, f, \hat{u}^\rho) \nu(dx) = \inf_{f \in \mathbb{C}^2} \sup_{x \in \mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho).$$

Proof. For $\kappa > 0$ small enough, the function $\int_{\mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho) \nu(dx)$ is convex in $f \in \mathcal{A}_\kappa$, is concave and upper semicontinuous in $\nu \in \mathcal{P}$, and is sup-compact in ν , the latter property being shown in analogy with the proof of Lemma 3.1. Invoking Theorem 7 on p.319 in Aubin and Ekeland [4],

$$\begin{aligned} \inf_{f \in \mathbb{C}^2} \sup_{x \in \mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho) &= \inf_{f \in \mathbb{C}^2 \cap \mathbb{C}_\ell^1} \sup_{\nu \in \mathcal{P}} \int_{\mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho) \nu(dx) \\ &\leq \inf_{f \in \mathcal{A}_\kappa} \sup_{\nu \in \mathcal{P}} \int_{\mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho) \nu(dx) = \sup_{\nu \in \mathcal{P}} \inf_{f \in \mathbb{C}^2 \cap \mathbb{C}_\ell^1} \int_{\mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho) \nu(dx) \\ &= \sup_{\nu \in \mathcal{P}} \inf_{f \in \mathbb{C}^2} \int_{\mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho) \nu(dx) \leq \inf_{f \in \mathbb{C}^2} \sup_{x \in \mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho). \end{aligned}$$

\square

Remark 3.8. One can also show that, if $\kappa > 0$ is small enough, then

$$\begin{aligned} F(\lambda) &= \sup_{\nu \in \mathcal{P}} \inf_{f \in \mathcal{A}_\kappa} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) = \inf_{f \in \mathcal{A}_\kappa} \sup_{\nu \in \mathcal{P}} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) \\ &= \sup_{\nu \in \mathcal{P}} \inf_{f \in \mathcal{A}_\kappa} \int_{\mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}) \nu(dx). \end{aligned}$$

4 Proofs of the main results

We prove Theorem 2.1 by proving, firstly, the upper bounds and, afterwards, the lower bounds.

4.1 The upper bounds

This subsection contains the proofs of (2.41) and (2.44). Let us note that, by (2.6),

$$\begin{aligned} L_t^\pi &= \int_0^1 M(\pi_s^t, X_s^t) ds + \frac{1}{\sqrt{t}} \int_0^1 N(\pi_s^t, X_s^t)^T dW_s^t \\ &= \int_0^1 \int_{\mathbb{R}^l} M(\pi_s^t, x) \mu^t(ds, dx) + \frac{1}{\sqrt{t}} \int_0^1 N(\pi_s^t, X_s^t)^T dW_s^t. \end{aligned} \quad (4.1)$$

4.1.1 The proof of (2.41).

By (2.1) and Itô's lemma, for \mathbb{C}^2 -function f ,

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t \nabla f(X_s)^T \theta(X_s) ds + \frac{1}{2} \int_0^t \text{tr}(\sigma(X_s) \sigma(X_s)^T \nabla^2 f(X_s)) ds \\ &\quad + \int_0^t \nabla f(X_s)^T \sigma(X_s) dW_s. \end{aligned}$$

Since the process $\exp(\int_0^t (\lambda N(\pi_s, X_s) + \nabla f(X_s)^T \sigma(X_s)) dW_s - (1/2) \int_0^t |\lambda N(\pi_s, X_s) + \nabla f(X_s) \sigma(X_s)|^2 ds)$ is a local martingale, where $\lambda \in \mathbb{R}$, by (2.1) and (4.1),

$$\begin{aligned} \mathbf{E} \exp(t\lambda L_t^\pi + f(X_t) - f(X_0) - t \int_0^1 \lambda M(\pi_s^t, X_s^t) ds - t \int_0^1 \nabla f(X_s^t)^T \theta(X_s^t) ds \\ - \frac{t}{2} \int_0^1 \text{tr}(\sigma(X_s^t) \sigma(X_s^t)^T \nabla^2 f(X_s^t)) ds - \frac{t}{2} \int_0^1 |\lambda N(\pi_s^t, X_s^t) + \sigma(X_s^t)^T \nabla f(X_s^t)|^2 ds) \leq 1. \end{aligned} \quad (4.2)$$

Let $\nu^t(dx) = \mu^t([0, 1], dx)$. By (2.16) and (2.24), for $\lambda \in [0, 1]$,

$$\mathbf{E} \exp(t\lambda L_t^\pi + f(X_t) - f(X_0) - t \int_{\mathbb{R}^l} H(x; \lambda, f) \nu^t(dx)) \leq 1. \quad (4.3)$$

Consequently,

$$\mathbf{E} \chi_{\{L_t^\pi \geq q\}} \exp(t\lambda L_t^\pi + f(X_t) - f(X_0) - t \int_{\mathbb{R}^l} H(x; \lambda, f) \nu^t(dx)) \leq 1$$

Thus,

$$\ln \mathbf{E} \chi_{\{L_t^\pi \geq q\}} e^{f(X_t) - f(X_0)} \leq \sup_{\nu \in \mathcal{P}} (-\lambda q t + t \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx)) = -\lambda q t + t \sup_{x \in \mathbb{R}^l} H(x; \lambda, f).$$

By the reverse Hölder inequality, for arbitrary $\epsilon > 0$,

$$\mathbf{E} \chi_{\{L_t^\pi \geq q\}} e^{f(X_t) - f(X_0)} \geq \mathbf{P}(L_t^\pi \geq q)^{1+\epsilon} (\mathbf{E} e^{-(f(X_t) - f(X_0))/\epsilon})^{-\epsilon},$$

so,

$$\frac{1+\epsilon}{t} \ln \mathbf{P}(L_t^\pi \geq q) \leq -\lambda q + \sup_{x \in \mathbb{R}^l} H(x; \lambda, f) + \frac{\epsilon}{t} \ln \mathbf{E} e^{-(f(X_t) - f(X_0))/\epsilon}.$$

We may assume that $\inf_{f \in \mathbb{C}^2} \sup_{x \in \mathbb{R}^l} H(x; \lambda, f) < \infty$. By Lemma 3.2, the latter infimum is attained at f^λ . Since, by hypotheses, $f^\lambda(x) \geq -C_1|x| - C_2$ for some positive C_1 and C_2 and $|X_0|$ is bounded, we have that

$$\limsup_{t \rightarrow \infty} \frac{1+\epsilon}{t} \ln \mathbf{P}(L_t^\pi \geq q) \leq -\lambda q + \inf_{f \in \mathbb{C}^2} \sup_{x \in \mathbb{R}^l} H(x; \lambda, f) + \limsup_{t \rightarrow \infty} \frac{\epsilon}{t} \ln \mathbf{E} e^{C_1|X_t|/\epsilon}.$$

Consequently, by $\mathbf{E} e^{C_1|X_t|/\epsilon}$ being bounded in t according to Lemma D.2 of the appendix and by ϵ being arbitrarily small,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L_t^\pi \geq q) \leq -(\lambda q - \inf_{f \in \mathbb{C}^2} \sup_{x \in \mathbb{R}^l} H(x; \lambda, f))$$

yielding (2.41), if one recalls (2.39b), (2.25), and F being convex so that the supremum in (2.39b) can be taken over $[0, 1]$.

4.1.2 The proof of (2.44)

Since $J_q^s = 0$ when $\hat{\lambda} \geq 0$, we may assume that $\hat{\lambda} < 0$. Letting $\pi_s^t = \hat{u}^\rho(X_s^t)$ in (4.2) yields, for $f \in \mathbb{C}^2$,

$$\mathbf{E} \exp(t\hat{\lambda} L_t^{\hat{\pi}^\rho} + f(X_t) - f(X_0) - t \int_{\mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho) \nu^t(dx)) \leq 1. \quad (4.4)$$

Therefore, on recalling that $\hat{\lambda} < 0$,

$$\begin{aligned} \mathbf{E} \mathbf{1}_{\{L^{\hat{\pi}^\rho} \leq q\}} \exp(f(X_t) - f(X_0)) &\leq e^{-t\hat{\lambda}q} \mathbf{E} \exp(t\hat{\lambda}L_t^{\hat{\pi}^\rho} + f(X_t) - f(X_0)) \\ &\leq e^{-t\hat{\lambda}q} \exp\left(t \sup_{\nu \in \mathcal{P}} \int_{\mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho) \nu(dx)\right). \end{aligned} \quad (4.5)$$

By the reverse Hölder inequality, for $\epsilon > 0$,

$$\mathbf{E} \mathbf{1}_{\{L^{\hat{\pi}^\rho} \leq q\}} \exp(f(X_t) - f(X_0)) \geq \mathbf{P}(L^{\hat{\pi}^\rho} \leq q)^{1+\epsilon} \mathbf{E} \exp(e^{-(1/\epsilon)(f(X_t) - f(X_0))})^{-\epsilon}. \quad (4.6)$$

Assuming that $f \in \mathcal{A}_\kappa$, with κ being small enough as compared with ϵ , we have, by (2.3), that

$$\limsup_{t \rightarrow \infty} \mathbf{E} \exp(e^{-(1/\epsilon)(f(X_t) - f(X_0))})^{1/t} \leq 1.$$

Therefore,

$$\limsup_{t \rightarrow \infty} \frac{1+\epsilon}{t} \ln \mathbf{P}(L^{\hat{\pi}^\rho} \leq q) \leq -\hat{\lambda}q + \inf_{f \in \mathcal{A}_\kappa} \sup_{\nu \in \mathcal{P}} \int_{\mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho) \nu(dx). \quad (4.7)$$

By Lemma 3.7 and (2.43),

$$\limsup_{\rho \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L^{\hat{\pi}^\rho} \leq q) \leq F(\hat{\lambda}).$$

4.2 The lower bounds

In this subsection, we prove (2.40) and (2.42). Let us assume that $\hat{\lambda} < \bar{\lambda}$. We prove that, if $q' > q$, then

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L_t^\pi < q') \geq -(\hat{\lambda}q - G(\hat{\lambda}, \hat{f}, \hat{m})) \quad (4.8a)$$

and that, if $q'' < q$, then

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L_t^{\hat{\pi}} > q'') \geq -(\hat{\lambda}q - G(\hat{\lambda}, \hat{f}, \hat{m})). \quad (4.8b)$$

We begin with showing that

$$\hat{\lambda}q - G(\hat{\lambda}, \hat{f}, \hat{m}) = \frac{1}{2} \int_{\mathbb{R}^l} |\hat{\lambda}N(\hat{u}(x), x) + \sigma(x)^T \nabla \hat{f}(x)|^2 \hat{m}(x) dx. \quad (4.9)$$

Since $(\hat{\lambda}, \hat{f}, \hat{m})$ is a saddle point of $\lambda q - \check{G}(\lambda, \nabla f, m)$ in $(-\infty, \bar{\lambda}] \times (\mathbb{C}^2 \cap \mathbb{C}_\ell^1) \times \mathbb{P}$ by Lemma 3.6, $\hat{\lambda}$ is the point of the maximum of the concave function $\lambda q - \check{G}(\lambda, \nabla \hat{f}, \hat{m})$ on $(-\infty, \bar{\lambda}]$.

Since $\hat{\lambda} < \bar{\lambda}$ and $\check{G}(\lambda, \nabla \hat{f}, \hat{m})$ is differentiable on $(-\infty, \bar{\lambda})$, the λ -derivative of $\check{G}(\lambda, \nabla \hat{f}, \hat{m})$ at $\hat{\lambda}$ equals zero. By (3.14) of Lemma 3.5,

$$\left. \frac{d}{d\lambda} \check{G}(\lambda, \nabla \hat{f}, \hat{m}) \right|_{\lambda=\hat{\lambda}} = \int_{\mathbb{R}^l} (M(\hat{u}(x), x) + \hat{\lambda} |N(\hat{u}(x), x)|^2 + \nabla \hat{f}(x)^T \sigma(x) N(\hat{u}(x), x)) \hat{m}(x) dx, \quad (4.10)$$

so,

$$\int_{\mathbb{R}^l} (M(\hat{u}(x), x) + \hat{\lambda} |N(\hat{u}(x), x)|^2 + \nabla \hat{f}(x)^T \sigma(x) N(\hat{u}(x), x)) \hat{m}(x) dx = q. \quad (4.11)$$

Therefore, by (2.16), (2.24), and (2.28),

$$\begin{aligned} \hat{\lambda} q - G(\hat{\lambda}, \hat{f}, \hat{m}) &= \hat{\lambda} \int_{\mathbb{R}^l} (M(\hat{u}(x), x) + \hat{\lambda} |N(\hat{u}(x), x)|^2 + \nabla \hat{f}(x)^T \sigma(x) N(\hat{u}(x), x)) \hat{m}(x) dx \\ &\quad - \int_{\mathbb{R}^l} \left(\hat{\lambda} M(\hat{u}(x), x) + \frac{1}{2} \hat{\lambda}^2 |N(\hat{u}(x), x)|^2 + \hat{\lambda} \nabla \hat{f}(x)^T \sigma(x) N(\hat{u}(x), x) \right. \\ &\quad \left. + \frac{1}{2} |\sigma(x)^T \nabla \hat{f}(x)|^2 + \nabla \hat{f}(x)^T \theta(x) + \frac{1}{2} \text{tr}(\sigma(x) \sigma(x)^T \nabla^2 \hat{f}(x)) \right) \hat{m}(x) dx \\ &= \int_{\mathbb{R}^l} \frac{1}{2} \hat{\lambda}^2 |N(\hat{u}(x), x)|^2 \hat{m}(x) dx - \int_{\mathbb{R}^l} \left(\frac{1}{2} |\sigma(x)^T \nabla \hat{f}(x)|^2 + \nabla \hat{f}(x)^T \theta(x) \right. \\ &\quad \left. + \frac{1}{2} \text{tr}(\sigma(x) \sigma(x)^T \nabla^2 \hat{f}(x)) \right) \hat{m}(x) dx. \end{aligned} \quad (4.12)$$

Integration by parts in (2.33) combined with the facts that $|\nabla \hat{f}(x)|$ grows at most linearly with $|x|$, that $\hat{u}(x)$ is a linear function of $\nabla \hat{f}(x)$ by (2.34), that $\int_{\mathbb{R}^l} |x|^2 \hat{m}(x) dx < \infty$, and that $\int_{\mathbb{R}^l} |\nabla \hat{m}(x)|^2 / \hat{m}(x) dx < \infty$, shows that (2.33) holds with $\hat{f}(x)$ as $h(x)$. Substitution on the rightmost side of (4.12) yields (4.9).

Let \hat{W}_s^t for $s \in [0, 1]$ and measure $\hat{\mathbf{P}}^t$ be defined by the respective equations

$$\hat{W}_s^t = W_s^t - \sqrt{t} \int_0^s (\hat{\lambda} N(\hat{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \hat{f}(X_s^t)) d\tilde{s} \quad (4.13)$$

and

$$\begin{aligned} \frac{d\hat{\mathbf{P}}^t}{d\mathbf{P}} &= \exp \left(\sqrt{t} \int_0^1 (\hat{\lambda} N(\hat{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \hat{f}(X_s^t))^T dW_s^t \right. \\ &\quad \left. - \frac{t}{2} \int_0^1 |\hat{\lambda} N(\hat{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \hat{f}(X_s^t)|^2 ds \right). \end{aligned} \quad (4.14)$$

A multidimensional extension of Theorem 4.7 on p.137 in Liptser and Shiryaev [27], which is proved similarly, obtains that, given $t > 0$, there exists $\gamma' > 0$ such that $\sup_{s \leq t} \mathbf{E} e^{\gamma' |X_s|^2} < \infty$.

By Example 3 on pp.220,221 in Liptser and Shiryaev [27] and the linear growth condition on $\nabla \hat{f}(x)$, the expectation of the righthand side of (4.14) with respect to \mathbf{P} equals unity. Therefore, $\hat{\mathbf{P}}^t$ is a valid probability measure and the process $(\hat{W}_s^t, s \in [0, 1])$ is a standard Wiener process under $\hat{\mathbf{P}}^t$, see Lemma 6.4 on p.216 in Liptser and Shiryaev [27] and Theorem 5.1 on p.191 in Karatzas and Shreve [23].

By (2.5b) and (2.34),

$$a(x) - r(x)\mathbf{1} + b(x)(\hat{\lambda}N(\hat{u}(x), x) + \sigma(x)^T \nabla \hat{f}(x)) = c(x)\hat{u}(x).$$

It follows that

$$\begin{aligned} L_t^\pi &= \int_0^1 M(\pi_s^t, X_s^t) ds + \frac{1}{\sqrt{t}} \int_0^1 N(\pi_s^t, X_s^t)^T dW_s^t = \int_0^1 M(\pi_s^t, X_s^t) ds \\ &\quad + \int_0^1 N(\pi_s^t, X_s^t)^T (\hat{\lambda}N(\hat{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \hat{f}(X_s^t)) ds + \frac{1}{\sqrt{t}} \int_0^1 N(\pi_s^t, X_s^t)^T d\hat{W}_s^t \\ &= \frac{1}{t} \ln \mathcal{E}_1^t + \int_0^1 M(\hat{u}(X_s^t), X_s^t) ds + \int_0^1 N(\hat{u}(X_s^t), X_s^t)^T (\hat{\lambda}N(\hat{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \hat{f}(X_s^t)) ds \\ &\quad + \frac{1}{\sqrt{t}} \int_0^1 N(\hat{u}(X_s^t), X_s^t)^T d\hat{W}_s^t, \quad (4.15) \end{aligned}$$

where \mathcal{E}_s^t represents the stochastic exponential defined by

$$\mathcal{E}_s^t = \exp\left(\sqrt{t} \int_0^s (\pi_s^t - \hat{u}(X_s^t))^T b(X_s^t) d\hat{W}_s^t - \frac{t}{2} \int_0^s \|\pi_s^t - \hat{u}(X_s^t)\|_{c(X_s^t)}^2 d\tilde{s}\right).$$

By (4.14) and (4.15), for $\delta > 0$,

$$\begin{aligned} \mathbf{P}(L_t^\pi < q + 3\delta) &= \hat{\mathbf{E}}^t \chi_{\left\{ \int_0^1 M(\pi_s^t, X_s^t) ds + \frac{1}{\sqrt{t}} \int_0^1 N(\pi_s^t, X_s^t)^T dW_s^t < q + 3\delta \right\}} \\ &\quad \exp\left(-\sqrt{t} \int_0^1 (\hat{\lambda}N(\hat{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \hat{f}(X_s^t))^T d\hat{W}_s^t \right. \\ &\quad \left. - \frac{t}{2} \int_0^1 |\hat{\lambda}N(\hat{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \hat{f}(X_s^t)|^2 ds \right) \\ &\geq \hat{\mathbf{E}}^t \chi_{\left\{ \frac{1}{t} \ln \mathcal{E}_1^t < \delta \right\}} \chi_{\left\{ \frac{1}{\sqrt{t}} \left| \int_0^1 N(\hat{u}(X_s^t), X_s^t)^T d\hat{W}_s^t \right| < \delta \right\}} \chi_{\left\{ \int_{\mathbb{R}^l} M(\hat{u}(x), x) \nu^t(dx) \right\}} \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}^l} N(\hat{u}(x), x)^T (\hat{\lambda} N(\hat{u}(x), x) + \sigma(x)^T \nabla \hat{f}(x)) \nu^t(dx) < q + \delta \Big\} \\
& \chi \Big\{ \frac{1}{\sqrt{t}} \Big| \int_0^1 (\hat{\lambda} N(\hat{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \hat{f}(X_s^t))^T d\hat{W}_s^t \Big| < \delta \Big\} \\
& \chi \Big\{ \int_{\mathbb{R}^L} |\hat{\lambda} N(\hat{u}(x), x) + \sigma(x)^T \nabla \hat{f}(x)|^2 \nu^t(dx) - \int_{\mathbb{R}^l} |\hat{\lambda} N(\hat{u}(x), x) + \sigma(x)^T \nabla \hat{f}(x)|^2 \hat{m}(x) dx < 2\delta \Big\} \\
& \exp\left(-2\delta t - \frac{t}{2} \int_{\mathbb{R}^l} |\hat{\lambda} N(\hat{u}(x), x) + \sigma(x)^T \nabla \hat{f}(x)|^2 \hat{m}(x) dx\right). \quad (4.16)
\end{aligned}$$

We will work with the terms on the righthand side in order. Since $\hat{\mathbf{E}}^t \mathcal{E}_1^t \leq 1$, Markov's inequality yields the convergence

$$\lim_{t \rightarrow \infty} \hat{\mathbf{P}}^t \left(\frac{1}{t} \ln \mathcal{E}_1^t < \delta \right) = 1. \quad (4.17)$$

By (2.1) and (4.13),

$$dX_s^t = t \theta(X_s^t) ds + t \sigma(X_s^t) \left(\hat{\lambda} N(\hat{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \hat{f}(X_s^t) \right) ds + \sqrt{t} \sigma(X_s^t) d\hat{W}_s^t.$$

Hence, the process $X = (X_s, s \geq 0) = (X_{s/t}^t, s \geq 0)$ satisfies the equation

$$dX_s = \theta(X_s) ds + \sigma(X_s) \left(\hat{\lambda} N(\hat{u}(X_s), X_s) + \sigma(X_s)^T \nabla \hat{f}(X_s) \right) ds + \sigma(X_s) d\tilde{W}_s^t,$$

(\tilde{W}_s^t) being a standard Wiener process under $\hat{\mathbf{P}}^t$. We note that by Theorem 10.1.3 on p.251 in Stroock and Varadhan [37] the distribution of X under $\hat{\mathbf{P}}^t$ is specified uniquely. In particular, it does not depend on t .

We show that if $g(x)$ is a continuous function such that $|g(x)| \leq K(1 + |x|^2)$, for all $x \in \mathbb{R}^l$ and some $K > 0$, then

$$\lim_{t \rightarrow \infty} \hat{\mathbf{P}}^t \left(\left| \int_{\mathbb{R}^l} g(x) \nu^t(dx) - \int_{\mathbb{R}^l} g(x) \hat{m}(x) dx \right| > \epsilon \right) = 0. \quad (4.18)$$

Since $\hat{m}(x)$ is a unique solution to (2.33), by Theorem 1.7.5 in Bogachev, Krylov, and Röckner [9], $\hat{m}(x) dx$ is a unique invariant measure of X under $\hat{\mathbf{P}}^t$, see also Proposition 9.2 on p.239 in Ethier and Kurtz [15]. It is thus an ergodic measure. We recall that $\hat{m} \in \hat{\mathbb{P}}$, so $\int_{\mathbb{R}^l} |x|^2 \hat{m}(x) dx < \infty$. Let P^* denote the probability measure on the space $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^l)$ of continuous \mathbb{R}^l -valued functions equipped with the locally uniform topology that is defined by $P^*(B) = \int_{\mathbb{R}^l} P_x(B) \hat{m}(x) dx$, where P_x is the distribution in $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^l)$ of process X started at x . Since $\hat{m}(x) dx$ is ergodic, so is P^* , see Corollary on p.12 in Skorokhod [36]. Hence, P^* -a.s.,

$$\lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s g(\tilde{X}_{\tilde{s}}) d\tilde{s} = \int_{\mathbb{R}^l} g(x) \hat{m}(x) dx, \quad (4.19)$$

see, e.g., Theorem 3 on p.9 in Skorokhod [36], with \tilde{X} representing a generic element of $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^l)$. Let \mathcal{C} denote the complement of the set of elements of $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^l)$ such that (4.19) holds. By Proposition 1.2.18 in Bogachev, Krylov, and Röckner [9], $\hat{m}(x)$ is continuous and strictly positive. Since $P^*(\mathcal{C}) = 0$, we have that $P_x(\mathcal{C}) = 0$ for almost all $x \in \mathbb{R}^l$ with respect to Lebesgue measure. It follows that if X_0 has an absolutely continuous distribution $n(x) dx$, then $\int_{\mathbb{R}^l} P_x(\mathcal{C}) n(x) dx = 0$, which means that (4.19) holds a.s. w.r.t. $\hat{\mathbf{P}}$, the latter symbol denoting the distribution of X on the space of trajectories. If the distribution of X_0 is not absolutely continuous, then the distribution of X_1 is because the transition probability has a density, see pp. 220–226 in Stroock and Varadhan [37]. Hence, (4.19) holds $\hat{\mathbf{P}}$ -a.s. for that case too. We have proved (4.18).

By (2.34), the linear growth condition on $\nabla \hat{f}(x)$, and (4.18),

$$\lim_{t \rightarrow \infty} \hat{\mathbf{P}}^t \left(\left| \int_{\mathbb{R}^l} |\hat{\lambda} N(\hat{u}(x), x) + \sigma(x)^T \nabla \hat{f}(x)|^2 \nu^t(dx) - \int_{\mathbb{R}^l} |\hat{\lambda} N(\hat{u}(x), x) + \sigma(x)^T \nabla \hat{f}(x)|^2 m(x) dx \right| < 2\delta \right) = 1. \quad (4.20)$$

Since, for $\eta > 0$, by the Lénglart–Rebolledo inequality, see Theorem 3 on p.66 in Liptser and Shirayev [28],

$$\begin{aligned} \hat{\mathbf{P}}^t \left(\left| \frac{1}{\sqrt{t}} \int_0^1 (\hat{\lambda} N(\hat{u}(X_s^t), X_s^t) + \sigma(x)^T \nabla \hat{f}(X_s^t)) d\hat{W}_s^t \right| \geq \delta \right) \\ \leq \frac{\eta}{\delta^2} + \hat{\mathbf{P}}^t \left(\int_0^1 |\hat{\lambda} N(\hat{u}(X_s^t), X_s^t) + \sigma(x)^T \nabla \hat{f}(X_s^t)|^2 ds \geq \eta t \right), \end{aligned}$$

we conclude that

$$\lim_{t \rightarrow \infty} \hat{\mathbf{P}}^t \left(\frac{1}{\sqrt{t}} \left| \int_0^1 (\hat{\lambda} N(\hat{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \hat{f}(X_s^t)) d\hat{W}_s^t \right| < \delta \right) = 1. \quad (4.21)$$

Similarly,

$$\lim_{t \rightarrow \infty} \hat{\mathbf{P}}^t \left(\frac{1}{\sqrt{t}} \left| \int_0^1 N(\hat{u}(X_s^t), X_s^t)^T d\hat{W}_s^t \right| < \delta \right) = 1. \quad (4.22)$$

By (4.11) and (4.18),

$$\lim_{t \rightarrow \infty} \hat{\mathbf{P}}^t \left(\int_{\mathbb{R}^l} (M(\hat{u}(x), x) + N(\hat{u}(x), x)^T (\hat{\lambda} N(\hat{u}(x), x) + \sigma(x)^T \nabla \hat{f}(x))) \nu^t(dx) < q + \delta \right) = 1.$$

Recalling (4.17) and (4.16) obtains that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L_t^\pi < q') \geq -\frac{1}{2} \int_{\mathbb{R}^l} |\hat{\lambda} N(\hat{u}(x), x) + \sigma(x)^T \nabla \hat{f}(x)|^2 m(x) dx, \quad (4.23)$$

so, (4.8a) follows from (4.9).

In order to prove (4.8b), we note that if $\pi_s^t = \hat{u}(X_s^t)$, then $\mathcal{E}_s^t = 0$ in (4.15), so

$$\begin{aligned} & \int_0^1 M(\hat{u}(X_s^t), X_s^t) ds + \frac{1}{\sqrt{t}} \int_0^1 N(\hat{u}(X_s^t), X_s^t)^T dW_s^t = \int_0^1 M(\hat{u}(X_s^t), X_s^t) ds \\ & + \int_0^1 N(\hat{u}(X_s^t), X_s^t)^T (\hat{\lambda}N(\hat{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \hat{f}(X_s^t)) ds + \frac{1}{\sqrt{t}} \int_0^1 N(\hat{u}(X_s^t), X_s^t)^T d\hat{W}_s^t. \end{aligned}$$

On recalling (4.1), similarly to (4.16),

$$\begin{aligned} \mathbf{P}(L_t^{\hat{\pi}} > q - 2\delta) &= \hat{\mathbf{E}}^t \chi \left\{ \int_0^1 \left(M(\hat{u}(X_s^t), X_s^t) + N(\hat{u}(X_s^t), X_s^t)^T (\hat{\lambda}N(\hat{u}(X_s^t), X_s^t) \right. \right. \\ & \quad \left. \left. + \sigma(X_s^t)^T \nabla \hat{f}(X_s^t)) \right) ds + \frac{1}{\sqrt{t}} \int_0^1 N(\hat{u}(X_s^t), X_s^t)^T d\hat{W}_s^t > q - 2\delta \right\} \\ & \exp\left(-\sqrt{t} \int_0^1 (\hat{\lambda}N(\hat{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \hat{f}(X_s^t))^T d\hat{W}_s^t + \frac{t}{2} \int_0^1 |\hat{\lambda}N(\hat{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \hat{f}(X_s^t)|^2 ds\right) \\ & \geq \chi \left\{ \frac{1}{\sqrt{t}} \int_0^1 N(\hat{u}(X_s^t), X_s^t)^T d\hat{W}_s^t > -\delta \right\} \chi \left\{ \int_{\mathbb{R}^l} \left(M(\hat{u}(x), x) \right. \right. \\ & \quad \left. \left. + N(\hat{u}(x), x)^T (\hat{\lambda}N(\hat{u}(x), x) + \sigma(x)^T \nabla \hat{f}(x)) \right) \nu^t(dx) \geq q - \delta \right\} \\ & \chi \left\{ \frac{1}{\sqrt{t}} \int_0^1 (\hat{\lambda}N(\hat{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \hat{f}(X_s^t))^T d\hat{W}_s^t \geq -\delta \right\} \\ & \chi \left\{ \int_{\mathbb{R}^l} |\hat{\lambda}N(\hat{u}(x), x) + \sigma(x)^T \nabla \hat{f}(x)|^2 \nu^t(dx) - \int_{\mathbb{R}^l} |\hat{\lambda}N(\hat{u}(x), x) + \sigma(x)^T \nabla \hat{f}(x)|^2 \hat{m}(x) dx \leq 2\delta \right\} \\ & \exp\left(-2\delta t - \frac{t}{2} \int_{\mathbb{R}^l} |\hat{\lambda}N(\hat{u}(x), x) + \sigma(x)^T \nabla \hat{f}(x)|^2 \hat{m}(x) dx\right). \quad (4.24) \end{aligned}$$

One still has (4.20), (4.21), and (4.22). By (4.11) and (4.18),

$$\lim_{t \rightarrow \infty} \hat{\mathbf{P}}^t \left(\int_{\mathbb{R}^l} (M(\hat{u}(x), x) + N(\hat{u}(x), x)^T (\hat{\lambda}N(\hat{u}(x), x) + \sigma(x)^T \nabla \hat{f}(x))) \nu^t(dx) > q - \delta \right) = 1.$$

Recalling (4.24) yields

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L_t^{\hat{\pi}} > q'') \geq -\frac{1}{2} \int_{\mathbb{R}^l} |\hat{\lambda}N(\hat{u}(x), x) + \sigma(x)^T \nabla \hat{f}(x)|^2 \hat{m}(x) dx, \quad (4.25)$$

so, (4.8b) follows from (4.9).

Reversing the roles of q and q' in (4.8a) and reversing the roles of q and q'' in (4.8b) obtain that, if $q' < q$, then

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L_t^\pi < q) \geq -J_{q'}^s$$

and that, if $q'' > q$, then

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L_t^{\hat{\pi}} > q) \geq -J_{q''}^o.$$

Letting $q' \rightarrow q$ and $q'' \rightarrow q$ and using the continuity of J_q^s and J_q^o , respectively, which properties hold by Lemma 3.5, prove (2.40) and (2.42), respectively, provided $\hat{\lambda} < \bar{\lambda}$.

Suppose that $\hat{\lambda} = \bar{\lambda} < 1$. Let $\hat{f} = f^{\hat{\lambda}}$ be as in Lemma 3.2. Then (4.23) and (4.25) hold by a similar argument to the one above. Since $\bar{\lambda}$ maximises $\lambda q - \check{G}(\lambda, \hat{f}, \hat{m})$ over λ we have that $(d/d\lambda) \check{G}(\lambda, \hat{f}, \hat{m})|_{\bar{\lambda}-} \leq q$. By (4.10) still holding, we have that in (4.11) the $=$ sign has to be replaced with \leq . By $\bar{\lambda}$ being positive, the first $=$ sign in (4.12) needs to be replaced with \geq , so does the $=$ sign in (4.9). By (4.23) and (4.25), one obtains (2.40) and (2.42), respectively.

Suppose that $\hat{\lambda} = \bar{\lambda} = 1$. Since $\hat{\lambda} > 0$, so, $J_q^s = 0$ and $J_q^o > 0$, (2.40) is a consequence of (2.41). We now work toward (2.42). Since 1 maximises $\lambda q - \check{F}(\lambda, \hat{m})$ over λ and the function $\check{F}(\lambda, \hat{m})$ is a convex function of λ , $\check{F}(1, \hat{m}) < \infty$ and $d/d\lambda \check{F}(\lambda, \hat{m})|_{1-} \leq q$. Let $\nabla \hat{f}$ be defined as in part 3 of Lemma 3.6, i.e., let $\inf_{\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \hat{m}(x) dx)} \check{G}(1, \nabla f, \hat{m})$ be attained at $\nabla \hat{f}$. By (3.16) of Lemma 3.5, $d/d\lambda \check{G}(\lambda, \nabla \hat{f}, \hat{m})|_{1-} \leq q$. By part 3 of Lemma 3.6, $\check{G}(1, \nabla \hat{f}, \hat{m})$ being finite implies that, $\hat{m}(x) dx$ -a.e.,

$$b(x)\sigma(x)^T \nabla \hat{f}(x) = b(x)\beta(x) - a(x) + r(x)\mathbf{1}. \quad (4.26)$$

By (3.14) of Lemma 3.5, if $\lambda < 1$, then

$$\frac{d\check{G}(\lambda, \nabla \hat{f}, \hat{m})}{d\lambda} = \int_{\mathbb{R}^l} (M(u^{\lambda, \nabla \hat{f}}(x), x) + \lambda |N(u^{\lambda, \nabla \hat{f}}(x), x)|^2 + N(u^{\lambda, \nabla \hat{f}}(x)^T \sigma(x)^T \nabla \hat{f}(x), x)) \hat{m}(x) dx,$$

where $u^{\lambda, \nabla \hat{f}}(x)$ is defined by (2.17) with $\nabla \hat{f}(x)$ as p . On noting that by (4.26) the limit, as $\lambda \uparrow 1$, in (2.17) with $\nabla \hat{f}(x)$ as p equals $c(x)^{-1}b(x)\beta$, we have, see Theorem 24.1 on p.227 in Rockafellar [35] for the first equality below, that

$$\begin{aligned} \frac{d}{d\lambda} \check{G}(\lambda, \nabla \hat{f}, \hat{m})|_{1-} &= \lim_{\lambda \uparrow 1} \frac{d}{d\lambda} \check{G}(\lambda, \nabla \hat{f}, \hat{m}) = \int_{\mathbb{R}^l} (M(c(x)^{-1}b(x)\beta(x), x) \\ &\quad + |N(c(x)^{-1}b(x)\beta(x), x)|^2 + N(c(x)^{-1}b(x)\beta(x), x)^T \sigma(x)^T \nabla \hat{f}(x)) \hat{m}(x) dx. \end{aligned}$$

We recall that $\hat{v}(x)$ is defined to be a bounded continuous function with values in the range of $b(x)^T$ such that $|\hat{v}(x)|^2/2 = q - d/d\lambda \check{F}(\lambda, \hat{m})|_{1-}$ and $\hat{u}(x) = c(x)^{-1}b(x)(\beta(x) +$

$\hat{v}(x)$). By Lemma 3.5, $d/d\lambda \check{F}(\lambda, \hat{m})|_{1-} = d/d\lambda \check{G}(\lambda, \nabla \hat{f}, \hat{m})|_{1-}$. Since the vectors $b(x)^T c(x)^{-1} b(x) \beta(x) - \beta(x)$ and $b(x)^T c(x)^{-1} b(x) \hat{v}(x)$ are orthogonal, with the former being in the null space of $b(x)$ and the latter being in the range of $b(x)^T$, substitution in (2.5a) and (2.5b) with the account of (2.22) yields

$$\begin{aligned} \int_{\mathbb{R}^l} (M(\hat{u}(x), x) + |N(\hat{u}(x), x)|^2 + N(\hat{u}(x), x)^T \sigma(x)^T \nabla \hat{f}(x)) \hat{m}(x) dx \\ = \frac{d}{d\lambda} \check{G}(\lambda, \nabla \hat{f}, \hat{m})|_{1-} + \int_{\mathbb{R}^l} \frac{|\hat{v}(x)|^2}{2} \hat{m}(x) dx = q. \end{aligned} \quad (4.27)$$

(As a consequence, (4.11) holds in this case too.)

We now invoke results in Puhalskii [33]. Let the process $\hat{\Psi}_t = (\hat{\Psi}_s^t, s \in [0, 1])$ be defined by (2.7) with $\hat{u}(x)$ as $u(x)$. Since $\hat{u}(x)$ is a bounded continuous function, the random variables $N(\hat{u}(X_s^t), X_s^t)$ are uniformly bounded. Condition 2.2 in Puhalskii [33] is fulfilled because part 2 of condition (N) implies that the length of the projection of $N(\hat{u}(x), x)$ onto the nullspace of $\sigma(x)$ is bounded away from zero and, consequently, the quantity $|N(\hat{u}(x), x)|^2 - N(\hat{u}(x), x)^T \sigma(x) (\sigma(x) \sigma(x)^T)^{-1} \sigma(x)^T N(\hat{u}(x), x)$ is bounded away from zero. Thus, Theorem 2.1 in Puhalskii [33] applies, so the pair $(\hat{\Psi}^t, \mu^t)$ satisfies the Large Deviation Principle in $\mathbb{C}([0, 1]) \times \mathbb{C}_\uparrow([0, 1], \mathbb{M}_1(\mathbb{R}^l))$ for rate t , as $t \rightarrow \infty$, with the deviation function in (2.9), provided the function $\Psi = (\Psi_s, s \in [0, 1])$ is absolutely continuous w.r.t. Lebesgue measure on \mathbb{R}_+ and the function $\mu = (\mu_s(\Gamma))$, when considered as a measure on $[0, 1] \times \mathbb{R}^l$, is absolutely continuous w.r.t. Lebesgue measure, i.e., $\mu(ds, dx) = m_s(x) dx ds$, where $m_s(x)$, as a function of x , belongs to $\hat{\mathbb{P}}$ for almost all s . If those conditions do not hold then $\mathbf{J}(\Psi, \mu) = \infty$. Since $L_t^{\hat{\pi}} = \hat{\Psi}_1^t$ and $\nu^t(\Gamma) = \mu^t([0, 1], \Gamma)$, by projection, the pair $(L_t^{\hat{\pi}}, \nu^t)$ obeys the Large Deviation Principle in $\mathbb{R} \times \mathbb{M}_1(\mathbb{R}^l)$ for rate t with deviation function $\mathbf{I}^{\hat{u}}$, such that $\mathbf{I}^{\hat{u}}(L, \nu) = \inf\{\mathbf{J}(\Psi, \mu) : \Psi_1 = L, \mu([0, 1], \Gamma) = \nu(\Gamma)\}$. Therefore,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L_t^{\hat{\pi}} > q) \geq - \inf_{(L, \nu): L > q} \mathbf{I}^{\hat{u}}(L, \nu). \quad (4.28)$$

Calculations show that

$$\mathbf{I}^{\hat{u}}(L, \nu) = \sup_{\lambda \in \mathbb{R}} (\lambda L - \inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} \overline{H}(x; \lambda, f, \hat{u}) \nu(dx)),$$

if $\nu(dx) = m(x) dx$, where $m \in \hat{\mathbb{P}}$, and $\mathbf{I}^{\hat{u}}(L, \nu) = \infty$, otherwise. By (2.37), the function $\lambda L - \inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} \overline{H}(x; \lambda, f, \hat{u}) \hat{m}(x) dx$ is concave in λ and is convex and lower semicontinuous in L . It is sup-compact in λ because $\mathbf{I}^{\hat{u}}(L, \nu)$ is a deviation function, i.e., it is inf-compact. (We provide a direct proof of the latter property in the appendix.) Therefore, by Theorem

7 on p.319 in Aubin and Ekeland [4],

$$\begin{aligned}
\inf_{(L,\nu): L>q} \mathbf{I}^{\hat{u}}(L,\nu) &\leq \inf_{L>q} \sup_{\lambda \in \mathbb{R}} (\lambda L - \inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} \overline{H}(x; \lambda, f, \hat{u}) \hat{m}(x) dx) \\
&= \sup_{\lambda \in \mathbb{R}} \inf_{L>q} (\lambda L - \inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} \overline{H}(x; \lambda, f, \hat{u}) \hat{m}(x) dx) = \sup_{\lambda \geq 0} (\lambda q - \inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} \overline{H}(x; \lambda, f, \hat{u}) \hat{m}(x) dx).
\end{aligned} \tag{4.29}$$

By integration by parts, if $f \in \mathbb{C}_0^2$, then, see (2.37),

$$\begin{aligned}
\int_{\mathbb{R}^l} \overline{H}(x; \lambda, f, v) \hat{m}(x) dx &= \int_{\mathbb{R}^l} (\lambda M(v(x), x) + \frac{1}{2} |\lambda N(v(x), x) + \sigma(x)^T \nabla f(x)|^2 + \nabla f(x)^T \theta(x) \\
&\quad - \frac{1}{2} \nabla f(x)^T \frac{\operatorname{div}(\sigma(x) \sigma(x)^T \hat{m}(x))}{\hat{m}(x)}) \hat{m}(x) dx. \tag{4.30}
\end{aligned}$$

As the righthand side depends on $f(x)$ through $\nabla f(x)$ only, similarly to developments above, we use the righthand side of (4.30) in order to define the lefthand side when $\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \hat{m}(x) dx)$. By the set of the gradients of \mathbb{C}_0^2 -functions being dense in $\mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \hat{m}(x) dx)$,

$$\inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} \overline{H}(x; \lambda, f, \hat{u}) \hat{m}(x) dx = \inf_{\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \hat{m}(x) dx)} \int_{\mathbb{R}^l} \overline{H}(x; \lambda, f, \hat{u}) \hat{m}(x) dx.$$

Since $\overline{H}(x; 1, f, \hat{u}) = H(x; 1, f)$ (see (2.21) and (4.26)), $\int_{\mathbb{R}^l} \overline{H}(x; 1, f, \hat{u}) \hat{m}(x) dx = \check{G}(1, \nabla f, \hat{m})$. By $\nabla \hat{f}$ minimising $\check{G}(1, \nabla f, \hat{m})$ over $\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \hat{m}(x) dx)$, the function $q - \int_{\mathbb{R}^l} \overline{H}(x; 1, f, \hat{u}) \hat{m}(x) dx$ attains maximum over ∇f in $\mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \hat{m}(x) dx)$ at $\nabla \hat{f}$. Therefore, the partial derivative with respect to ∇f of $\lambda q - \int_{\mathbb{R}^l} \overline{H}(x; \lambda, f, \hat{u}) \hat{m}(x) dx$ equals zero at $(1, \nabla \hat{f})$. By (4.30), we can write (4.27) as $d/d\lambda \int_{\mathbb{R}^l} \overline{H}(x; \lambda, \hat{f}, \hat{u}) \hat{m}(x) dx \Big|_1 = q$, so, the partial derivative with respect to λ of $\lambda q - \int_{\mathbb{R}^l} \overline{H}(x; \lambda, f, \hat{u}) \hat{m}(x) dx$ at $(1, \nabla \hat{f})$ equals zero too. The function $\lambda q - \int_{\mathbb{R}^l} \overline{H}(x; \lambda, f, \hat{u}) \hat{m}(x) dx$ being concave in $(\lambda, \nabla f)$, it therefore attains a global maximum in $\mathbb{R} \times \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \hat{m}(x) dx)$ at $(1, \nabla \hat{f})$, cf. Proposition 1.2 on p.36 in Ekeland and Temam [14]. Hence,

$$\sup_{\lambda \geq 0} (\lambda q - \inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} \overline{H}(x; \lambda, f, \hat{u}) \hat{m}(x) dx) = q - \check{G}(1, \nabla \hat{f}, \hat{m}).$$

The latter expression being equal to J_q° , (4.28), and (4.29) imply the required lower bound (2.42).

5 The proof of Theorem 2.2

For the first assertion of part 1, let us assume that $\lambda < \bar{\lambda}$. Let $\epsilon > 0$ be such that $\lambda(1+\epsilon) < \bar{\lambda}$. Let f_ϵ represent the function $f^{\lambda(1+\epsilon)}$. By (2.16), (2.24), (2.25), and (4.3),

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{E} \exp((1+\epsilon)\lambda t L_t^\pi + f_\epsilon(X_t) - f_\epsilon(X_0)) \leq F((1+\epsilon)\lambda). \quad (5.1)$$

By the reverse Hölder inequality,

$$\mathbf{E} \exp((1+\epsilon)\lambda t L_t^\pi + f_\epsilon(X_t) - f_\epsilon(X_0)) \geq (\mathbf{E} \exp(\lambda t L_t^\pi))^{1+\epsilon} (\mathbf{E} \exp(-(1/\epsilon)(f_\epsilon(X_t) - f_\epsilon(X_0))))^{-\epsilon},$$

so, since f_ϵ is bounded below by an affine function and $|X_0|$ is bounded, in analogy with the proof of (2.41),

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{E} \exp(\lambda t L_t^\pi) \leq F(\lambda).$$

The latter inequality is trivially true if $\lambda > \bar{\lambda}$.

We address the lower bound. Let $0 < \lambda < \bar{\lambda}$. Then F is subdifferentiable at λ . Let q represent a subgradient of F at λ . Since $\lambda q - F(\lambda) = J_q^\circ$, by (2.42),

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{E} e^{\lambda t L_t^{\pi^\lambda}} &\geq \liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{E} e^{\lambda t L_t^{\pi^\lambda}} \chi_{\{L_t^{\pi^\lambda} \geq q\}} \geq \lambda q + \liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L_t^{\pi^\lambda} \geq q) \\ &\geq \lambda q - J_q^\circ = F(\lambda). \end{aligned} \quad (5.2)$$

If $\lambda = \bar{\lambda}$ and F is subdifferentiable at $\bar{\lambda}$, a similar proof applies. Suppose that $\lambda = \bar{\lambda}$ and F is not subdifferentiable at $\bar{\lambda}$. By what has been just proved,

$$\liminf_{\check{\lambda} \uparrow \bar{\lambda}} \liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{E} e^{\check{\lambda} t L_t^{\pi^{\check{\lambda}}}} \geq \liminf_{\check{\lambda} \uparrow \bar{\lambda}} F(\check{\lambda}) = F(\bar{\lambda})$$

and Hölder's inequality yields

$$\liminf_{\check{\lambda} \uparrow \bar{\lambda}} \liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{E} e^{\bar{\lambda} t L_t^{\pi^{\check{\lambda}}}} \geq F(\bar{\lambda}).$$

By requiring $\pi_t^{\bar{\lambda}}$ to match $\pi_t^{\lambda_i}$ on certain intervals $[t_i, t_{i+1})$ where $\lambda_i \uparrow \bar{\lambda}$ and $t_i \rightarrow \infty$ appropriately, we can ensure that $\liminf_{t \rightarrow \infty} (1/t) \ln \mathbf{E} e^{\bar{\lambda} t L_t^{\pi^{\bar{\lambda}}}} \geq F(\bar{\lambda})$.

Suppose that $\lambda > \bar{\lambda}$. If F is subdifferentiable at $\bar{\lambda}$, then, similarly to (5.2), on choosing q as a subgradient of F at $\bar{\lambda}$,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{E} e^{\lambda t L_t^{\pi^{\bar{\lambda}}}} \geq \lambda q + \liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L_t^{\pi^{\bar{\lambda}}} \geq q) \geq \lambda q - J_q^\circ = (\lambda - \bar{\lambda})q + F(\bar{\lambda}). \quad (5.3)$$

Since q can be chosen arbitrarily great, $\lim_{t \rightarrow \infty} (1/t) \ln \mathbf{E} e^{\lambda t L_t^{\pi^{\bar{\lambda}}}} = \infty$. If F is not subdifferentiable at $\bar{\lambda}$, then we pick λ_i and q_i such that $\lambda_i \uparrow \bar{\lambda}$, q_i is a subgradient of F at λ_i and $q_i \uparrow \infty$. Arguing along the lines of (5.3) yields

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{E} e^{\lambda t L_t^{\pi^{\lambda_i}}} \geq \lambda q_i + \liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(L_t^{\pi^{\lambda_i}} \geq q_i) \geq (\lambda - \bar{\lambda})q_i + F(\lambda_i),$$

so there exists π^λ such that $\lim_{t \rightarrow \infty} (1/t) \ln \mathbf{E} e^{\lambda t L_t^{\pi^\lambda}} = \infty$.

We prove now part 2. Since $\mathbf{E} e^{\lambda t L_t^\pi} \geq e^{\lambda q t} \mathbf{P}(L_t^\pi \leq q)$ provided $\lambda < 0$, the inequality in (2.40) of Theorem 2.1 implies that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{E} e^{\lambda t L_t^\pi} \geq \sup_{q \in \mathbb{R}} (\lambda q - J_q^s) = F(\lambda),$$

with the latter equality holding because by (2.39c) J_q^s is the Legendre–Fenchel transform of the function that equals $F(\lambda)$ when $\lambda \leq 0$ and equals ∞ , otherwise.

Since $\lambda < 0$, F is differentiable at λ , so π^λ is well defined. Let $u^\lambda(x)$ be such that $\pi_t^\lambda = u^\lambda(X_t)$, i.e., $u^\lambda(x)$ is defined as $\hat{u}(x)$ when $q = F'(\lambda)$. By (4.2), assuming that $f \in \mathcal{A}_\kappa$,

$$\mathbf{E} \exp(\lambda t L_t^{\pi^{\lambda, \rho}} + f(X_t) - f(X_0) - t \int_{\mathbb{R}^l} \overline{H}(x; \lambda, f, u^{\lambda, \rho}) \nu^t(dx)) \leq 1.$$

By Lemma 3.7, recalling that $|X_0|$ is bounded,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{E} \exp(\lambda t L_t^{\pi^{\lambda, \rho}}) \leq \inf_{f \in \mathbb{C}^2} \sup_{x \in \mathbb{R}^l} \overline{H}(x; \lambda, f, u^{\lambda, \rho}).$$

We now apply condition (2.43).

A The scalar case

We will assume that $l = n = 1$, so, in (2.47a)–(2.47e), $\Theta_1, \theta_2, A_1, a_2, r_1, r_2, \alpha_1$, and α_2 are scalars, $\Theta_1 < 0$, σ is a $1 \times k$ -matrix, b is a $1 \times k$ -matrix, and β is a k -vector. Accordingly, $c, \sigma \sigma^T, \sigma b^T, P_1(\lambda), p_2(\lambda), A(\lambda), B(\lambda)$, and C are scalars. The equation for $P_1(\lambda)$ is

$$B(\lambda) P_1(\lambda)^2 + 2A(\lambda) P_1(\lambda) + \frac{\lambda}{1 - \lambda} C = 0. \quad (\text{A.1})$$

Let

$$\tilde{\beta} = 1 + \frac{1}{\Theta_1^2} \frac{A_1 - r_1}{c} (\sigma \sigma^T (A_1 - r_1) - 2\Theta_1 \sigma b^T). \quad (\text{A.2})$$

(The latter piece of notation is modelled on that of Pham [31].) We have that

$$A(\lambda)^2 - B(\lambda) \frac{\lambda}{1 - \lambda} C = \Theta_1^2 \frac{1 - \lambda \tilde{\beta}}{1 - \lambda}.$$

Hence, $P_1(\lambda)$ exists if and only if

$$\lambda \leq \frac{1}{\tilde{\beta}} \wedge 1,$$

so, $\tilde{\lambda} = \min(1/\tilde{\beta}, 1)$. (Not unexpectedly, if $\lambda < 0$ then (A.1) has both a positive and a negative root, whereas both roots are positive if $0 < \lambda \leq \tilde{\lambda}$.) If $\lambda < \tilde{\lambda}$, then

$$P_1(\lambda) = \frac{1}{B(\lambda)} \left(-A(\lambda) - |\Theta_1| \sqrt{\frac{1 - \lambda \tilde{\beta}}{1 - \lambda}} \right) \quad (\text{A.3})$$

and $F(\lambda)$ is determined by (2.50) and (2.53). The minus sign in front of the square root is chosen because $D(\lambda) = A(\lambda) + B(\lambda)P_1(\lambda)$ has to be negative which is needed in order for the analogue of (2.52) to have a stationary distribution. Therefore,

$$D(\lambda) = \Theta_1 \sqrt{\frac{1 - \lambda\tilde{\beta}}{1 - \lambda}}. \quad (\text{A.4})$$

The functions $D(\lambda)$ and $P_1(\lambda)$ are differentiable for $\lambda < 1 \wedge (1/\tilde{\beta})$. As in Pham [31], we distinguish between three cases: $\tilde{\beta} > 1$, $\tilde{\beta} < 1$, and $\tilde{\beta} = 1$.

Suppose that $\tilde{\beta} > 1$ so, $\tilde{\lambda} = 1/\tilde{\beta}$. Then $P_1(\lambda)$ and $D(\lambda)$ are continuous on $[0, 1/\tilde{\beta}]$ and differentiable on $(0, 1/\tilde{\beta})$. We have that $P_1(1/\tilde{\beta}) = -A(1/\tilde{\beta})/B(1/\tilde{\beta})$ and $D(1/\tilde{\beta}) = 0$. Also, $D(\lambda)/\sqrt{1/\tilde{\beta} - \lambda} \rightarrow -|\Theta_1|\sqrt{\tilde{\beta}}/\sqrt{1 - 1/\tilde{\beta}}$ and $(P_1(1/\tilde{\beta}) - P_1(\lambda))/\sqrt{1/\tilde{\beta} - \lambda} \rightarrow |\Theta_1|\sqrt{\tilde{\beta}}/(B(1/\tilde{\beta})\sqrt{1 - 1/\tilde{\beta}})$, as $\lambda \uparrow 1/\tilde{\beta}$. In addition, by (2.50) and (2.53), if $E(1/\tilde{\beta}) \neq 0$, then $|p_2(\lambda)| = |E(\lambda)/D(\lambda)| \rightarrow \infty$ and $F(\lambda) \rightarrow \infty$, so, $F(\lambda) = \infty$ when $\lambda \geq 1/\tilde{\beta}$, $\bar{\lambda} = 1/\tilde{\beta}$, and $\hat{\lambda} < \bar{\lambda}$. Suppose that $E(1/\tilde{\beta}) = 0$. By (2.50) and (2.51), $E(\lambda) = D(\lambda)Z(\lambda) + U(\lambda)$, where

$$Z(\lambda) = \frac{\lambda}{1 - \lambda} b\sigma^T c^{-1}(a_2 - r_2 - \lambda b\beta) - \lambda\sigma\beta + \theta_2$$

and

$$U(\lambda) = \frac{\lambda}{1 - \lambda} (A_1 - r_1)c^{-1}(a_2 - r_2 - \lambda b\beta) + \lambda(r_1 - \alpha_1) - \frac{A(\lambda)}{B(\lambda)} Z(\lambda).$$

Therefore,

$$p_2(\lambda) = -\frac{Z(\lambda)}{B(\lambda)} - \frac{U(\lambda)}{D(\lambda)},$$

Since $E(1/\tilde{\beta}) = D(1/\tilde{\beta}) = 0$, $U(1/\tilde{\beta}) = 0$. By $U(\lambda)$ being linear in a neighbourhood of $1/\tilde{\beta}$, $p_2(\lambda)$ is continuous at $1/\tilde{\beta}$, $p_2(1/\tilde{\beta}) = -Z(1/\tilde{\beta})/B(1/\tilde{\beta})$, and $F(1/\tilde{\beta})$ is finite.

Let us look at the derivative at $1/\tilde{\beta}$. We have that $(p_2(1/\tilde{\beta}) - p_2(\lambda))/\sqrt{1/\tilde{\beta} - \lambda} \rightarrow U'(1/\tilde{\beta})\sqrt{1 - 1/\tilde{\beta}}/(\Theta_1\sqrt{\tilde{\beta}})$, as $\lambda \uparrow 1/\tilde{\beta}$. By (2.53), $(F(1/\tilde{\beta}) - F(\lambda))/\sqrt{1/\tilde{\beta} - \lambda} \rightarrow (1/2)\sigma\sigma^T|\Theta_1|\sqrt{\tilde{\beta}}/(B(1/\tilde{\beta})\sqrt{1 - 1/\tilde{\beta}})$. Therefore, $F'(1/\tilde{\beta}-) = \infty$, so, $\bar{\lambda} = 1/\tilde{\beta}$ and $\hat{\lambda} < \bar{\lambda}$.

Suppose that $\tilde{\beta} < 1$. By (A.2), $b\sigma^T \neq 0$. Also, $\tilde{\lambda} = \bar{\lambda} = 1$. By (A.3) and (A.4), $P_1(\lambda)$ has limit $P_1(1)$ when $\lambda \uparrow 1$ and $(P_1(\lambda) - P_1(1))/\sqrt{1 - \lambda} \rightarrow \Theta_1\sqrt{1 - \tilde{\beta}}/((b\sigma^T)^2c^{-1})$ as $\lambda \uparrow 1$. In fact, $P_1(1) = -(A_1 - r_1)/(b\sigma^T)$. By (A.4), (2.50), (2.51), and (2.53), $p_2(\lambda) \rightarrow -(a_2 - r_2 - b\beta)/b\sigma^T$, as $\lambda \uparrow 1$, which quantity we denote by $p_2(1)$. By (2.50) and (2.51), on noting that $A_1 - r_1 + b\sigma^T P_1(1) = 0$.

$$\lim_{\lambda \uparrow 1} \frac{p_2(1) - p_2(\lambda)}{\sqrt{1 - \lambda}} = K_1, \quad (\text{A.5})$$

where

$$K_1 = \frac{1}{\Theta_1\sqrt{1 - \tilde{\beta}}} \left(\left(\Theta_1 - \frac{\sigma\sigma^T(A_1 - r_1)}{b\sigma^T} \right) p_2(1) + r_1 - \alpha_1 + P_1(1)(\theta_2 - \sigma\beta) \right).$$

Since $a_2 - r_2 - b\beta + b\sigma^T p_2(1) = 0$,

$$\lim_{\lambda \uparrow 1} \frac{a_2 - r_2 - \lambda b\beta + b\sigma^T p_2(\lambda)}{\sqrt{1 - \lambda}} = \lim_{\lambda \uparrow 1} \frac{b\sigma^T (p_2(\lambda) - p_2(1))}{\sqrt{1 - \lambda}} = b\sigma^T K_1.$$

By (2.51), $F(1-) < \infty$. Let us look at the derivative $F'(1-)$. One needs to improve on (A.5). More specifically, by (2.50), (2.51), (A.3) and (A.4), one can expand as follows (either by hand or by the use of Mathematica): as $\lambda \uparrow 1$,

$$p_2(\lambda) = p_2(1) - K_1 \sqrt{1 - \lambda} - K_2(1 - \lambda) + o(1 - \lambda),$$

where

$$K_2 = \frac{\sigma\sigma^T}{(b\sigma^T)^2 c^{-1}} p_2(1) + \frac{b\beta}{b\sigma^T} + \frac{\theta_2 - \sigma\beta}{(b\sigma^T)^2 c^{-1}}.$$

By (2.53),

$$\lim_{\lambda \uparrow 1} \frac{F(\lambda) - F(1)}{\sqrt{1 - \lambda}} = -\sigma\sigma^T p_2(1) K_1 - b\sigma^T K_1 (b\beta - b\sigma^T K_2) c^{-1} + (\sigma\beta - \theta_2) K_1 + \frac{1}{2} \sigma\sigma^T \frac{\Theta_1 \sqrt{1 - \tilde{\beta}}}{(b\sigma^T)^2 c^{-1}},$$

which simplifies to

$$\lim_{\lambda \uparrow 1} \frac{F(1) - F(\lambda)}{\sqrt{1 - \lambda}} = \frac{|\Theta_1| \sqrt{1 - \tilde{\beta}} \sigma\sigma^T}{2(b\sigma^T)^2 c^{-1}},$$

implying that $F'(1-) = \infty$, so, $\hat{\lambda} < \bar{\lambda}$.

Let us consider the case that $\tilde{\beta} = 1$, so, $(A_1 - r_1)(\sigma\sigma^T(A_1 - r_1) - 2\Theta_1 \sigma b^T) = 0$. One has that $\tilde{\lambda} = \bar{\lambda} = 1$, $D(\lambda) = \Theta_1$, $P_1(\lambda) = (-\sigma b^T c^{-1}(A_1 - r_1)) / ((1 - \lambda)/\lambda \sigma\sigma^T + \sigma b^T c^{-1} b\sigma^T)$ and $p_2(\lambda) = -E(\lambda)/\Theta_1$. Thus, if $b\sigma^T = 0$, then $A_1 - r_1 = 0$ and $P_1(\lambda) = 0$. If $b\sigma^T \neq 0$, then $P_1(1) = -(A_1 - r_1)/(b\sigma^T)$, $P_1'(1) = -\sigma\sigma^T(A_1 - r_1)/((b\sigma^T)^3 c^{-1})$, and $P_1''(1) = 2\sigma\sigma^T(A_1 - r_1)/((b\sigma^T)^3 c^{-1})(1 - \sigma\sigma^T/((b\sigma^T)^2 c^{-1}))$. Since

$$A_1 - r_1 + b\sigma^T P_1(1) = 0, \tag{A.6}$$

$E(\lambda)$ is continuous on $[0, 1]$ and is differentiable on $(0, 1)$, see (2.51), so is $p_2(\lambda)$. By (2.53), if $a_2 - r_2 - b\beta + b\sigma^T p_2(1) \neq 0$, then $F(\lambda) \rightarrow \infty$, as $\lambda \rightarrow \infty$, so $\hat{\lambda} < \bar{\lambda}$. If

$$a_2 - r_2 - b\beta + b\sigma^T p_2(1) = 0, \tag{A.7}$$

then

$$F(1) = \frac{1}{2} \sigma\sigma^T p_2(1)^2 + (-\sigma\beta + \theta_2) p_2(1) + r_2 - \alpha_2 + |\beta|^2 + \frac{1}{2} \sigma\sigma^T P_1(1)$$

and

$$\begin{aligned} F'(1-) &= \sigma\sigma^T p_2'(1-) p_2(1) + \frac{1}{2c} (b\sigma^T p_2'(1-) - b\beta)^2 - \beta^T \sigma^T p_2(1) + (-\sigma\beta + \theta_2) p_2'(1-) \\ &\quad + r_2 - \alpha_2 + \frac{3}{2} |\beta|^2 + \frac{1}{2} \sigma\sigma^T P_1'(1-). \end{aligned}$$

As one can see, $F(\lambda)$ is not essentially smooth. We obtain that $\hat{\lambda} < \bar{\lambda}$ if and only if $F'(1-) > q$, otherwise $\hat{\lambda} = 1$. It is noteworthy that (A.6) and (A.7) represent conditions (2.54a) and (2.54b), respectively.

The cases where $\tilde{\beta} \geq 1$ and $F(\lambda) \rightarrow \infty$ as $\lambda \uparrow 1/\tilde{\beta}$ and where $\tilde{\beta} < 1$ have been analysed by Pham [31].

B Proof of Lemma 2.1

Suppose that the matrix $\sigma(x)Q_1(x)\sigma(x)^T$ is uniformly positive definite. Then $|Q_1(x)\sigma(x)^Ty| \geq k_1|y|$, for some $k_1 > 0$, all $x \in \mathbb{R}^l$ and all $y \in \mathbb{R}^k$. Since $|\sigma(x)^Ty|^2 = y^T\sigma(x)\sigma(x)^Ty \leq k_2|y|^2$, for some $k_2 \geq k_1$, we have that

$$\frac{|(I_k - Q_1(x))\sigma(x)^Ty|}{|\sigma(x)^Ty|} \leq \frac{\sqrt{|\sigma(x)^Ty|^2 - k_1^2|y|^2}}{|\sigma(x)^Ty|} \leq \sqrt{1 - \frac{k_1^2}{k_2^2}}.$$

Therefore, since $I_k - Q_1(x)$ is the operator of the orthogonal projection on the range of $b(x)^T$, given $z \in \mathbb{R}^n$,

$$(\sigma(x)^Ty)^T b(x)^T z \leq \sqrt{1 - \frac{k_1^2}{k_2^2}} |\sigma(x)^Ty| |b(x)^T z|,$$

so nonzero vectors from the ranges of $\sigma(x)^T$ and of $b(x)^T$ are at angles uniformly bounded away from zero. Conversely, if $(\sigma(x)^Ty)^T b(x)^T z \leq \rho_1 |\sigma(x)^Ty| |b(x)^T z|$, for some $\rho_1 \in (0, 1)$, then $|(I_k - Q_1(x))\sigma(x)^Ty| \leq \rho_1 |\sigma(x)^Ty|$ so that $|Q_1(x)\sigma(x)^Ty| = \sqrt{|\sigma(x)^Ty|^2 - |(I_k - Q_1(x))\sigma(x)^Ty|^2} \geq (1 - \rho_1) |\sigma(x)^Ty| \geq (1 - \rho_1) \rho_2 |y|$, the latter inequality holding by $\sigma(x)\sigma(x)^T$ being uniformly positive definite, where $\rho_2 > 0$. Thus, the matrix $\sigma(x)Q_1(x)\sigma(x)^T$ is uniformly positive definite if and only if "the angle condition" holds. Since the angle condition is symmetric in $\sigma(x)$ and $b(x)$, it is also equivalent to the matrix $c(x) - b(x)\sigma(x)^T(\sigma(x)\sigma(x)^T)^{-1}\sigma(x)b(x)^T$ being uniformly positive definite.

In order to prove the second assertion of the lemma, let us observe that

$$\beta(x)^T Q_2(x) \beta(x) = \beta(x)^T Q_1(x) (I_k - Q_1(x) \sigma(x)^T (\sigma(x) Q_1(x) Q_1(x) \sigma(x)^T)^{-1} \sigma(x) Q_1(x)) Q_1(x) \beta(x),$$

so, if $\beta(x)^T Q_2(x) \beta(x)$ is bounded away from zero, then, by $|\beta(x) Q_1(x)|$ being bounded, there exists $\rho_3 \in (0, 1)$ such that, for all $x \in \mathbb{R}^l$,

$$(1 - \rho_3) |Q_1(x) \beta(x)| > (Q_1(x) \sigma(x)^T (\sigma(x) Q_1(x) Q_1(x) \sigma(x)^T)^{-1} \sigma(x) Q_1(x)) Q_1(x) \beta(x).$$

The righthand side representing the orthogonal projection of $Q_1(x) \beta(x)$ onto the range of $(\sigma(x) Q_1(x))^T$ implies that, given $y \in \mathbb{R}^l$,

$$|(Q_1(x) \beta(x))^T Q_1(x) \sigma(x)^T y| \leq \rho_3 |Q_1(x) \beta(x)| |Q_1(x) \sigma(x)^T y|,$$

which means that $Q_1(x) \beta(x)$ is at angles to $Q_1(x) \sigma(x)^T y$ which are bounded below uniformly over y . The converse is proved similarly.

C Proof of Lemma 2.2

By Lemma 3.7,

$$\inf_{f \in \mathbb{C}^2} \sup_{x \in \mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho) = \sup_{\nu \in \mathcal{P}} \inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho) \nu(dx). \quad (\text{C.1})$$

For function f and $\rho > 0$, we denote $f(x)^\rho = f(x)\chi_{[0,\rho]}(|x|)$. By (2.5a), (2.5b), (2.34), and (2.23),

$$\begin{aligned}
& \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho) \\
&= -\frac{\hat{\lambda}}{2(1-\hat{\lambda})} (\|b(x)\sigma(x)^T \nabla \hat{f}(x)^\rho\|_{c(x)^{-1}}^2 - \|(a(x) - r(x)\mathbf{1})^\rho\|_{c(x)^{-1}}^2) \\
&\quad + \hat{\lambda}(r(x) - \alpha(x) + \frac{1}{2}|\beta(x)|^2) + \frac{\hat{\lambda}}{2(1-\hat{\lambda})} \|\hat{\lambda}b(x)\beta(x)^\rho\|_{c(x)^{-1}}^2 \\
&\quad + \frac{\hat{\lambda}}{1-\hat{\lambda}} \left(-((a(x) - r(x)\mathbf{1})^\rho)^T c(x)^{-1} b(x) \hat{\lambda} \beta(x) \right. \\
&\quad \left. + ((a(x) - r(x)\mathbf{1} - \hat{\lambda}b(x)\beta(x) + b(x)\sigma(x)^T \nabla \hat{f}(x)^\rho)^T c(x)^{-1} b(x) \sigma(x)^T \nabla f(x)) \right. \\
&\quad \left. + \frac{1}{2} |-\hat{\lambda}\beta(x) + \sigma(x)^T \nabla f(x)|^2 + \nabla f(x)^T \theta(x) + \frac{1}{2} \text{tr}(\sigma(x)\sigma(x)^T \nabla^2 f(x)) \right). \quad (\text{C.2})
\end{aligned}$$

As in the proof of Lemma 3.1, it follows that, under the hypotheses, there exist $\bar{\kappa} > 0$, $\bar{K}_1 > 0$ and $\bar{K}_2 > 0$ such that $\overline{H}(x; \hat{\lambda}, f_{\bar{\kappa}}, \hat{u}^\rho) \leq \bar{K}_1 - \bar{K}_2|x|^2$, for all $x \in \mathbb{R}^l$ and all $\rho > 0$. Consequently, $\inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho) \nu(dx)$ is a sup-compact function of $\nu \in \mathcal{P}$, so, the supremum over ν on the righthand side of (C.1) is attained at some ν_ρ . Moreover, if the limsup on the lefthand side of (2.43) is greater than $-\infty$, then

$$\limsup_{\rho \rightarrow \infty} \int_{\mathbb{R}^l} |x|^2 \nu_\rho(dx) < \infty, \quad (\text{C.3})$$

so, the ν_ρ make up a relatively compact subset of \mathcal{P} .

If (2.45) holds, then, given $\tilde{f} \in \mathbb{C}_0^2$, by (C.2), there exist \tilde{C}_1 and \tilde{C}_2 , such that, for all $x \in \mathbb{R}^l$ and all $\rho > 0$,

$$\overline{H}(x; \hat{\lambda}, \tilde{f}, \hat{u}^\rho) \leq \tilde{C}_1|x| + \tilde{C}_2. \quad (\text{C.4})$$

Assuming that $\nu_\rho \rightarrow \tilde{\nu}$, we have, by the convergence $\overline{H}(x_\rho; \hat{\lambda}, \tilde{f}, \hat{u}^\rho) \rightarrow \overline{H}(\tilde{x}; \hat{\lambda}, \tilde{f}, \hat{u})$ when $x_\rho \rightarrow \tilde{x}$, by (C.3), (C.4), the definition of the topology on \mathcal{P} , Fatou's lemma, and the dominated convergence theorem, that

$$\limsup_{\rho \rightarrow \infty} \int_{\mathbb{R}^l} \overline{H}(x; \hat{\lambda}, \tilde{f}, \hat{u}^\rho) \nu_\rho(dx) \leq \int_{\mathbb{R}^l} \overline{H}(x; \hat{\lambda}, \tilde{f}, \hat{u}) \tilde{\nu}(dx),$$

so, on recalling (2.38),

$$\limsup_{\rho \rightarrow \infty} \inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}^\rho) \nu_\rho(dx) \leq \inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} \overline{H}(x; \hat{\lambda}, f, \hat{u}) \tilde{\nu}(dx) \leq F(\hat{\lambda}).$$

D

Lemma D.1. *Given $L \in \mathbb{R}$, $m \in \hat{\mathbb{P}}$, and $v \in \mathbb{L}^2(\mathbb{R}^l, \mathbb{R}^n, m(x) dx)$, the sets*

$$\{\lambda \in \mathbb{R} : \lambda L - \inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} \overline{H}(x; \lambda, f, v) m(x) dx \geq \alpha\}$$

are compact for all $\alpha \in \mathbb{R}$.

Proof. By (2.37),

$$\begin{aligned} \inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} \overline{H}(x; \lambda, f, v) m(x) dx &= \inf_{f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)} \int_{\mathbb{R}^l} (\lambda M(v(x), x) \\ &+ \frac{1}{2} |\lambda N(v(x), x) + \sigma(x)^T \nabla f(x)|^2 + \nabla f(x)^T \theta(x) - \frac{1}{2} \nabla f(x)^T \frac{\text{div}(\sigma(x) \sigma(x)^T m(x))}{m(x)}) m(x) dx. \end{aligned}$$

The infimum is attained at

$$\nabla f(x) = \lambda g_1(x) + g_2(x),$$

where

$$\begin{aligned} g_1 &= \Pi((\sigma(\cdot) \sigma(\cdot)^T)^{-1} \sigma(\cdot)^T N(v(\cdot), \cdot)), \\ g_2 &= \Pi((\sigma(\cdot) \sigma(\cdot)^T)^{-1} (-\theta(\cdot) + \nabla f(\cdot)^T \frac{\text{div}(\sigma(\cdot) \sigma(\cdot)^T m(\cdot))}{2m(\cdot)})), \end{aligned}$$

with Π representing the operator of the orthogonal projection on $\mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)$ in $\mathbb{L}^2(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)$ with respect to the inner product $\langle h_1, h_2 \rangle = \int_{\mathbb{R}^l} h_1(x)^T \sigma(x) \sigma(x)^T h_2(x) m(x) dx$. Therefore,

$$\begin{aligned} \lambda L - \inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} \overline{H}(x; \lambda, f, v) m(x) dx \\ &= \lambda (L - \int_{\mathbb{R}^l} M(v(x), x) m(x) dx - \int_{\mathbb{R}^l} g_1(x)^T \sigma(x) \sigma(x)^T g_1(x) m(x) dx) \\ &- \frac{1}{2} \int_{\mathbb{R}^l} g_2(x)^T \sigma(x) \sigma(x)^T g_2(x) m(x) dx - \frac{\lambda^2}{2} \int_{\mathbb{R}^l} (|N(v(x), x)|^2 - g_1(x)^T \sigma(x) \sigma(x)^T g_1(x)) m(x) dx. \end{aligned} \tag{D.1}$$

Since projection is a contraction operator,

$$\int_{\mathbb{R}^l} g_1(x)^T \sigma(x) \sigma(x)^T g_1(x) m(x) dx \leq \int_{\mathbb{R}^l} N(v(x), x)^T \sigma(x)^T (\sigma(x) \sigma(x)^T)^{-1} \sigma(x) N(v(x), x) m(x) dx.$$

As mentioned, by condition (N), $\beta(x)$ does not belong to the sum of the ranges of $b(x)^T$ and of $\sigma(x)^T$. By (2.5b), $N(u, x)$ does not belong to the range of $\sigma(x)^T$, for any u and x . Therefore, the projection of $N(v(x), x)$ onto the null space of $\sigma(x)$ is nonzero which implies that $|N(v(x), x)|^2 - N(v(x), x)^T \sigma(x)^T (\sigma(x) \sigma(x)^T)^{-1} \sigma(x) N(v(x), x)$ is positive for any x , so, the coefficient of λ^2 on the righthand side of (D.1) is positive, yielding the needed property. \square

The next result seems to be "well known". We haven't been able to find a reference, though.

Lemma D.2. For arbitrary $\kappa > 0$,

$$\limsup_{t \rightarrow \infty} \mathbf{E} e^{\kappa |X_t|} < \infty.$$

Proof. We employ Veretennikov's reflection device, see, e.g., Veretennikov [39]. Let us assume, at first, that $\sigma(x)\sigma(x)^T$ is the $l \times l$ identity matrix so that (2.1) can be written as

$$dX_t = \theta(X_t) dt + d\tilde{W}_t, \quad (\text{D.2})$$

with $\tilde{W} = (\tilde{W}_t, t \geq 0)$ being a standard \mathbb{R}^l -valued Wiener process. Let $K > 0$ and $R > 0$ be such that

$$x^T \theta(x) < -K|x|^2, \text{ provided } |x| \geq R.$$

By (D.2) and Itô's lemma, cf. Theorem 7.1 on p.218 in Karatzas and Shreve [23],

$$d(|X_t| \vee R) = \frac{X_t^T}{|X_t|} \theta(X_t) \chi_{(R, \infty)}(|X_t|) dt + \frac{X_t^T}{|X_t|} \chi_{(R, \infty)}(|X_t|) d\tilde{W}_t + d\phi_t, \quad (\text{D.3})$$

with ϕ_t being nondecreasing, continuous, $\phi_0 = 0$, and $\phi_t = \int_0^t \chi_R(|X_s|) d\phi_s$, where we let $X_t/|X_t| = \tilde{z}$, a fixed unit length vector from \mathbb{R}^l , should $|X_t| = 0$. Let us note that $(\int_0^t (X_s^T/|X_s|) d\tilde{W}_s, t \geq 0)$ is a one-dimensional standard Wiener process.

Let $\tilde{X} = (\tilde{X}_t, t \geq 0)$ represent the process that solves the equation

$$d\tilde{X}_t = -K \tilde{X}_t dt + \frac{X_t^T}{|X_t|} d\tilde{W}_t + d\tilde{\phi}_t,$$

where $\tilde{X}_0 = |X_0| \vee R$, $\tilde{X}_t \geq R$ and $\tilde{\phi}_t$ is nondecreasing, continuous, $\tilde{\phi}_0 = 0$, and $\tilde{\phi}_t = \int_0^t \chi_R(\tilde{X}_s) d\tilde{\phi}_s$. Thus, \tilde{X} is the scalar process that is the Skorohod reflection of $(-K \tilde{X}_t + \int_0^t X_s^T/|X_s| d\tilde{W}_s, t \geq 0)$ upwards at level R . Let us show that, a.s.,

$$\tilde{X}_t \geq |X_t| \vee R. \quad (\text{D.4})$$

Since

$$\begin{aligned} d(|X_t| \vee R - \tilde{X}_t) &= \left(\frac{X_t^T}{|X_t|} \theta(X_t) \chi_{(R, \infty)}(|X_t|) + K \tilde{X}_t \right) dt - \frac{X_t^T}{|X_t|} \chi_{(-\infty, R]}(|X_t|) d\tilde{W}_t \\ &\quad + d\phi_t - d\tilde{\phi}_t, \end{aligned}$$

by Itô's lemma, on taking into account that $\tilde{X}_t \geq R$, with $a^+ = a \vee 0$,

$$\begin{aligned} d\left((|X_t| \vee R - \tilde{X}_t)^+\right)^2 &= 2(|X_t| \vee R - \tilde{X}_t)^+ \chi_{\{|X_t| \vee R - \tilde{X}_t > 0\}} d(|X_t| \vee R - \tilde{X}_t) \\ &\quad + \chi_{\{|X_t| \vee R - \tilde{X}_t > 0\}} \chi_{(-\infty, R]}(|X_t|) dt = 2(|X_t| - \tilde{X}_t)^+ \left(\frac{X_t^T}{|X_t|} \theta(X_t) \chi_{(R, \infty)}(|X_t|) + K \tilde{X}_t \right) dt \\ &\quad + 2(|X_t| \vee R - \tilde{X}_t)^+ d(\phi_t - \tilde{\phi}_t), \end{aligned}$$

If $|X_t| \geq R$, then $(X_t^T/|X_t|)\theta(X_t) \leq -K|X_t|$, so,

$$(|X_t| - \tilde{X}_t)^+ \left(\frac{X_t^T}{|X_t|} \theta(X_t) \chi_{(R, \infty)}(|X_t|) + K \tilde{X}_t \right) \leq K(|X_t| - \tilde{X}_t)^+ (-|X_t| \chi_{(R, \infty)}(|X_t|) + \tilde{X}_t) \leq 0.$$

Since ϕ_t grows only when $|X_t| = R$ and $\tilde{X}_t \geq R$, $(|X_t| \vee R - \tilde{X}_t)^+ d\phi_t = 0$. We conclude that $((|X_t| \vee R - \tilde{X}_t)^+)^2 \leq 0$, which implies (D.4).

Let

$$d\check{X}_t = -K \check{X}_t dt + d\tilde{W}_t, \check{X}_0 = X_0. \quad (\text{D.5})$$

We have that

$$d(|\check{X}_t| \vee R) = -K |\check{X}_t| \chi_{(R, \infty)}(|\check{X}_t|) dt + \chi_{(R, \infty)}(|\check{X}_t|) \frac{\check{X}_t^T}{|\check{X}_t|} d\tilde{W}_t + \check{\phi}_t,$$

where $\check{\phi}_t$ is nondecreasing, continuous, $\check{\phi}_0 = 0$, and $\check{\phi}_t = \int_0^t \chi_R(|\check{X}_s|) d\check{\phi}_s$. Let

$$\tau_t = \inf \left\{ u : \int_0^u \chi_{(R, \infty)}(|\check{X}_s|) ds > t \right\}.$$

Then, with $\hat{X}_t = |\check{X}_{\tau_t}|$, we have that $\hat{X}_t \geq R$ and

$$d\hat{X}_t = -K \hat{X}_t dt + d\hat{W}_t + d\hat{\phi}_t, \quad (\text{D.6})$$

where $\hat{W} = (\hat{W}_t, t \geq 0)$ is a standard Wiener process, $\hat{\phi}_t$ is nondecreasing, continuous, $\hat{\phi}_0 = 0$, and $\hat{\phi}_t = \int_0^t \chi_R(\hat{X}_s) d\hat{\phi}_s$. The processes $(\int_0^t X_s^T/|X_s| d\tilde{W}_s, t \geq 0)$ and \hat{W} being identically distributed and $\hat{X}_0 = |X_0| \vee R$, we have, by the weak uniqueness of a solution to (D.6), that \tilde{X} and $\hat{X} = (\hat{X}_t, t \geq 0)$ are identically distributed.

By (D.5),

$$\check{X}_t = e^{-Kt} \left(X_0 + \int_0^t e^{Ks} d\tilde{W}_s \right).$$

Therefore, for $\lambda \in \mathbb{R}^l$ and stopping time τ with respect to the filtration generated by (\check{X}_t) ,

$$\begin{aligned} \mathbf{E} e^{\kappa \lambda^T \check{X}_\tau} &\leq (\mathbf{E} e^{2e^{-K\tau} \lambda^T X_0})^{1/2} (\mathbf{E} e^{2e^{-K\tau} \int_0^\tau e^{Ks} \lambda^T d\tilde{W}_s})^{1/2} \\ &\leq (\mathbf{E} e^{2e^{-K\tau} \lambda^T X_0})^{1/2} (\mathbf{E} e^{2e^{-K\tau} \int_0^\tau e^{Ks} \lambda^T d\tilde{W}_s})^{1/2} \leq (\mathbf{E} e^{2e^{-K\tau} \lambda^T X_0})^{1/2} (\mathbf{E} e^{4e^{-2K\tau} \int_0^\tau e^{2Ks} |\lambda|^2 ds})^{1/4} \\ &\leq (\mathbf{E} e^{2|\lambda| |X_0|})^{1/2} e^{1/(8K)}. \end{aligned}$$

Since $\mathbf{E} e^{\kappa |\check{X}_\tau|} \leq l \max_{\lambda: |\lambda|=1} \mathbf{E} e^{\kappa \sqrt{l} \lambda^T \check{X}_\tau}$, we have that $\sup_\tau \mathbf{E} e^{\kappa |\check{X}_\tau|} < \infty$. Hence, on noting that τ_t is a stopping time, $\sup_{t \geq 0} \mathbf{E} e^{\kappa |\hat{X}_t|} = \sup_{t \geq 0} \mathbf{E} e^{\kappa |\check{X}_{\tau_t}|} < \infty$, so that, by (D.4) and by \hat{X} and \tilde{X} being identically distributed, $\sup_{t \geq 0} \mathbf{E} e^{\kappa |X_t|} < \infty$.

Let us suppose now that $\sigma(x)\sigma(x)^T$ is not necessarily the identity matrix. Let

$$\bar{\tau}_t = \inf \left\{ u \geq 0 : \int_0^u |\sigma(X_s) \frac{X_s}{|X_s|}|^2 ds > t \right\}$$

and $\overline{X}_t = X_{\overline{\tau}_t}$. Then

$$d\overline{X}_t = \frac{\theta(\overline{X}_t)}{|\sigma(\overline{X}_t)\overline{X}_t/|\overline{X}_t||^2} dt + \frac{\sigma(\overline{X}_t)}{|\sigma(\overline{X}_t)\overline{X}_t/|\overline{X}_t||} d\overline{W}_t,$$

$\overline{W} = (\overline{W}_t, t \geq 0)$ being a standard Wiener process. Therefore,

$$\begin{aligned} d(|\overline{X}_t| \vee R) &= \frac{\overline{X}_t^T}{|\overline{X}_t|} \frac{\theta(\overline{X}_t)}{|\sigma(\overline{X}_t)\overline{X}_t/|\overline{X}_t||^2} \chi_{(R,\infty)}(|\overline{X}_t|) dt \\ &\quad + \frac{\overline{X}_t^T}{|\overline{X}_t|} \frac{\sigma(\overline{X}_t)}{|\sigma(\overline{X}_t)\overline{X}_t/|\overline{X}_t||} \chi_{(R,\infty)}(|\overline{X}_t|) d\tilde{W}_t + d\overline{\phi}_t, \end{aligned} \quad (\text{D.7})$$

where $\overline{\phi}_t$ is nondecreasing, continuous, $\overline{\phi}_0 = 0$, and $\overline{\phi}_t = \int_0^t \chi_R(|X_s|) d\overline{\phi}_s$. The process $(\int_0^t (\overline{X}_s^T/|\overline{X}_s|)\sigma(\overline{X}_s)/|\sigma(\overline{X}_s)\overline{X}_s/|\overline{X}_s|| d\tilde{W}_s, t \geq 0)$ being a one-dimensional standard Wiener process, (D.7) is analogous to (D.3). A similar line of reasoning to the one used above implies that $\sup_{\overline{\tau}} \mathbf{E}e^{\kappa|\overline{X}_{\overline{\tau}}|} < \infty$, where $\overline{\tau}$ represents a stopping time relative to the filtration generated by $(\overline{X}_t, t \geq 0)$. Since $X_t = \overline{X}_{\overline{\tau}_t^{-1}}$, we conclude that $\sup_{t \geq 0} \mathbf{E}e^{\kappa|X_t|} < \infty$, where $\overline{\tau}_t^{-1} = \inf\{u \geq 0 : \int_0^u |\sigma(\overline{X}_s)\overline{X}_s/|\overline{X}_s||^2 ds > t\}$. \square

Remark D.1. As the proof shows, $\sup_{\tau} \mathbf{E}e^{\kappa|X_{\tau}|} < \infty$, with the supremum being taken over all stopping times τ .

Remark D.2. An argument in Veretennikov [38] that we could have referred to seems to be flawed.

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